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Unstable invariant manifolds for stochastic PDEs driven by a fractional Brownian motion ☆

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In this paper, we consider a class of stochastic partial differential equations (SPDEs) driven by a fractional Brownian motion (fBm) with the Hurst parameter bigger than $1/2$. The existence of local random unstable manifolds is shown if the linear parts of these SPDEs are hyperbolic. For this purpose we introduce a modified Lyapunov–Perron transform, which contains stochastic integrals. By the singularities inside these integrals we obtain a special Lyapunov–Perron's approach by treating a segment of the solution over time interval $[0, 1]$ as a starting point and setting up an infinite series equation involving these segments as time evolves. Using this approach, we establish the existence of local random unstable manifolds in a tempered neighborhood of an equilibrium.

1. Introduction

In this paper, we study the qualitative properties of a stochastic evolution equation in a separable Hilbert space V

$$\begin{cases} du(t) = \big(Au(t) + F\big(u(t)\big)\big)\,dt + G\big(u(t)\big)\,dB^H(t), \\ u(0) = u_0 \in V, \end{cases} \tag{1.1}$$

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where A is the infinitesimal generator of an analytic semigroup, F and G are nonlinear terms, and B^H is the infinite dimensional fractional Brownian motion (fBm) with the Hurst parameter $H \in (1/2, 1)$.

An fBm appears naturally in the modeling of many complex phenomena in applications when the systems are subject to “rough” external forcing. An fBm is a stochastic process which deviates significantly from the standard Brownian motion and semi-martingales, and others classically used in the theory of stochastic process. As a centered Gaussian process, it is characterized by the stationarity of its increments and a medium- or long-memory property. It also exhibits power scaling and path regularity properties with the Hölder parameter. However, an fBm is not a semi-martingale nor a Markov process.

Since the pioneering works of Zähle [36], Decreusefond and Üstünel [10], and Lyons [23], the main thrust has been to understand how to perform stochastic integration with respect to an fBm in a way which is consistent with some properties of the classical Ito theory for Brownian motion. In the case of higher regularity ($H > 1/2$), the trajectory methods, labelled as pathwise, can be used, which make it easier to translate one integration theory into another, as the fractional derivatives allow a pathwise estimate of the integrals in terms of integrand and integrator using special norms. The pathwise integrals historically gave the first cases where adequate solutions to SPDEs were established; e.g. Nualart and Răşcanu [28]; infinite dimensional equations have been treated with the same success as finite dimensional ones, e.g. Nualart and Maslowski [24], Garrido-Atienza et al. [14], Tindel et al. [34], and Gubinelli et al. [16].

A fundamental problem in the study of dynamics of a stochastic partial differential equation is to show that it generates a random dynamical system (or stochastic flow). It is well known that a large class of partial differential equations with stationary random coefficients and Ito stochastic ordinary differential equations generate random dynamical systems (see Arnold [1]). However, for the stochastic partial differential equations driven by the standard Brownian motion, the problem is much more difficult. The reasons are: (i) The stochastic integral is only defined almost surely where the exceptional set may depend on the initial state; and (ii) Kolmogorov’s theorem, as cited in Kunita [20, Theorem 1.4.1], is only true for finite dimensional random fields. However, there are some partial results for additive as well as multiplicative noise (see for example, [13,11,12,6,27]). In the recent work [14], under appropriate conditions on A, F and G, it has been shown that the stochastic partial differential equation (1.1) generates a random dynamical system when $H \in (1/2, 1)$. One interesting fact is that, in contrast to most of papers that exist in the literature, in [14] the term G is a non-trivial diffusion coefficient. The approach in that paper is based on the stochastic calculus for an fBm introduced by Zähle.

This paper is devoted to the invariant manifolds for the random dynamical system generated by Eq. (1.1). The theory of invariant manifolds provides indispensable tools for the study of dynamics of nonlinear systems in finite or infinite dimensional spaces. Invariant manifolds can be used to capture complex dynamics and the long term behavior of solutions and to reduce high dimensional problems to the analysis of lower dimensional structures. Invariant manifolds provide a coordinate system in which systems of differential equations may be decoupled and normal forms derived. These play an important role in the study of structural stability of dynamical systems or, when a degeneracy occurs, in understanding the nature of bifurcations.

The two basic strategies for obtaining invariant manifolds were introduced by Hadamard [17] and Lyapunov and Perron (see [22] and [30,31]). Hadamard’s method is a geometric construction based on the graph transforms by using invariant cones, while Lyapunov–Perron approach is an analytic construction based on an integral equation using the exponential dichotomy. Both methods require the estimates of solutions in the stable and unstable directions. One can apply these methods either directly to the continuous time dynamical system or firstly to the time-one map (the discrete-time system) and then show that these invariant manifolds indeed are invariant manifolds for the continuous time systems. See, for example, [2], where the infinite dimensional deterministic dynamical systems were considered. In many cases, working first with discrete time systems does make computations easier.

There is a trade-off by using the stochastic calculus for an fBm in which the stochastic integral is defined by the generalized Stieltjes integral. On one hand, as it has been proved in [14], it allows us to show that Eq. (1.1) generates a random dynamical system by avoiding the obstacle caused by the invalidation of Kolmogorov's theorem in infinite dimensional space. On the other hand, it causes blowup for the estimates of orbits in stable direction for both the continuous time system and the time-one map because this type of stochastic integrals involves singularities in integrands. To overcome the difficult, we treat a segment of the solution over time interval $[0, 1]$ as a starting point unlike the usual approaches and we set up an infinite series equation involving these segments as time evolves. This approach is at the middle of the continuous time approach and the time-one map approach. Here, we will first prove a global unstable manifold theorem for a version of our equation with cut-off over a random neighborhood for Eq. (1.1). Then, we will construct a local unstable manifold of a stationary solution of Eq. (1.1). It is known that the existence of invariant manifolds depends on the size of perturbations and the spectral gap of the linear part of equations. The size of perturbations we have here may grow at a subexponential rate along the random base flow. Thus, we will be able to obtain only invariant manifolds in a tempered neighborhood of an equilibrium. Since the size of a tempered neighborhood can shrink only at a subexponential rate, the solutions on invariant manifolds will stay in the tempered neighborhood for a relatively long period of time, especially for the unstable manifold the backward orbits will stay in the neighborhood for all past time.

We would like to point out that the method is different when dealing with the existence of stable invariant manifolds, and this topic will be considered in a forthcoming paper.

The rich history of development of the theory of invariant manifolds for deterministic dynamical systems dates back to work of Hadamard, Perron, and Lyapunov, and includes notable advances due to Androsov, Bogoliubov, Chow, Fenichel, Hale, Hartman, Henry, Hirsch, Pugh and Shub, Krylov, Kurzweil, Levinson, Mañé, Marsden, Pliss, Ruelle, Sacker, Sell, and many others, too numerous to list here. There are works on invariant manifolds for stochastic or random ordinary differential equations (finite dimensional systems), see, for example, [35,1,26,32]. For stochastic partial differential equations, due to their non-classical fluctuation of driving noise and infinite dimensionality, the theory of invariant manifold is still in its infancy. Recently, there have been some works on invariant manifolds for stochastic partial differential equations with either additive noise or multiplicative noise, see [3,11,12,4,27] for pathwise random invariant manifolds and see [8] and [9] for almost surely invariant manifolds. The existence of invariant manifolds for stochastic wave equations with nonlinear multiplicative noise was proved in [21].

We organize this paper as follows. In Section 2, we introduce basic concepts on random dynamical systems. In Section 3, we first introduce the stochastic calculus for an fBm based on the generalized Stieltjes integral by Zähle and then we establish sufficient conditions on A, F and G in order to ensure that Eq. (1.1) has a unique solution that generates a random dynamical system. We introduce the Lyapunov–Perron transform for the segments of functions in Section 4 and discuss the properties of this transform. In Section 5, we prove the existence of local random unstable manifolds for the original equations. Finally, Section 6 contains an example as application of the developed theory.

2. Random dynamical systems

In this section we review some basic concepts and results on random dynamical systems that will be used to analyze the dynamics of SPDEs driven by a fractional Brownian motion.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $V = (V, \|\cdot\|, (\cdot,\cdot))$ be a separable Hilbert space.

Definition 2.1. A mapping

$$\Omega \ni \omega \to D(\omega) \in \mathcal{P}(V) \setminus \{\emptyset\}^1$$

is called a random set if $D(\omega)$ is a closed set and for every $y \in V$ the mapping

$$\omega \to \operatorname{dist}\big(y, D(\omega)\big) := \inf_{d \in D(\omega)} \|y - d\|$$

[1] $\mathcal{P}(V)$ is the power set of V.

is a random variable. A random variable d is called a selector of the random set D if $d(\omega) \in D(\omega)$ for all $\omega \in \Omega$.

D is a random set if and only if there exists a countable number of selectors $(d_i)_{i\in\mathbb{N}}$ such that

$$D(\omega) = \overline{\bigcup_{i\in\mathbb{N}} d_i(\omega)} \quad \text{for all } \omega \in \Omega,$$

see [7].

In the next definition, we introduce a system that models the evolution of a noise.

Definition 2.2. A metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with two-sided $\mathbb{T}$ (which is $\mathbb{R}$ in the continuous case and $\mathbb{Z}$ in the discrete one) consists of a measurable flow

$$\theta : \big(\mathbb{T} \times \Omega, \mathcal{B}(\mathbb{T}) \otimes \mathcal{F}\big) \to (\Omega, \mathcal{F}),$$

where the *flow* property for the mapping θ holds for the partial mappings $\theta_t = \theta(t, \cdot)$:

$$\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1}\theta_{t_2} = \theta_{t_1+t_2}, \qquad \theta_0 = \mathrm{id}_\Omega$$

for all $t_1, t_2 \in \mathbb{T}$, and $\mathbb{P}$ is supposed to be θ-ergodic.

The next concept is of fundamental importance in the study of random dynamical systems.

Definition 2.3. (i) A random variable $x \in (0, \infty)$ is called tempered with respect to a metric dynamical system θ if

$$\lim_{\mathbb{T}\ni t\to\pm\infty} \frac{1}{t} \log x(\theta_t\omega) = 0 \quad \mathbb{P}\text{-almost surely.} \tag{2.1}$$

(ii) $x : \Omega \to [0, \infty)$ is called tempered from above if

$$\lim_{\mathbb{T}\ni t\to\pm\infty} \frac{1}{t} \log^+ x(\theta_t\omega) = 0 \quad \mathbb{P}\text{-almost surely.}$$

(iii) $x : \Omega \to (0, \infty)$ is called tempered from below if $1/x$ is tempered from above.

The temperedness reflexes the subexponential growth of the mapping $t \to x(\theta_t\omega)$. In the ergodic case, the only alternative property is that

$$\limsup_{\mathbb{T}\ni t\to\pm\infty} \frac{\log x(\theta_t\omega)}{t} = \infty \quad \mathbb{P}\text{-almost surely.}$$

Thus, if x has a finite exponential growth, then x is tempered from above. Furthermore, if $\log x \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{T} = \mathbb{Z}$ then (2.1) holds on a $\{\theta_t\}_{t\in\mathbb{T}}$-invariant set of full $\mathbb{P}$-measure. If

$$\sup_{t\in[0,1]} \log x(\theta_t\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$$

then the same is true for $\mathbb{T} = \mathbb{R}$. For details we refer to Arnold [1, p. 164].

Let Ω' be a $\{\theta_t\}_{t\in\mathbb{T}}$-invariant set of full $\mathbb{P}$-measure, $\mathcal{F}'$ be the trace-σ-algebra of $\mathcal{F}$ with respect to Ω', $\mathbb{P}'$ be the restriction of $\mathbb{P}$ to $\mathcal{F}'$, and θ' be the restriction of θ to $\mathbb{T}\times\Omega'$. Then θ' is $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F}',\mathcal{F}')$-measurable, see Caraballo et al. [5], such that $(\Omega',\mathcal{F}',\mathbb{P}',\theta')$ is a metric dynamical system. We say *a property holds θ-almost surely* if this property holds for *every* ω in a $\{\theta_t\}_{t\in\mathbb{T}}$-invariant set of full $\mathbb{P}$-measure. In this case we will change the metric dynamical system in the above way but with the convention that we use the old notation for the new metric dynamical system. For instance, we will frequently use that a tempered from above random variable has a subexponential growth θ-almost surely.

We now introduce as an example of metric dynamical systems a special noise that is called *fractional Brownian motion*.

Given $H\in(0,1)$, a continuous centered Gaussian process $\beta^H(t)$, $t\in\mathbb{R}$, with the covariance function

$$\mathbb{E}\beta^H(t)\beta^H(s)=\frac{1}{2}\big(|t|^{2H}+|s|^{2H}-|t-s|^{2H}\big),\quad t,s\in\mathbb{R},$$

is called a *two-sided one-dimensional fractional Brownian motion* (fBm), and H is the *Hurst parameter*.

Assume that Q is a bounded and symmetric linear operator on V which is of trace class, i.e., there exist a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in V and a sequence of nonnegative numbers $\{\lambda_i\}_{i\in\mathbb{N}}$ such that $\operatorname{tr} Q:=\sum_{i=1}^\infty\lambda_i<\infty$ and $Qe_i=\lambda_i e_i$, $i\in\mathbb{N}$. Then a continuous *V-valued fractional Brownian motion* B^H with incremental covariance operator Q and Hurst parameter H is defined by

$$B^H(t)=\sum_{i=1}^\infty\sqrt{\lambda_i}e_i\beta_i^H(t),\quad t\in\mathbb{R},$$

where $\{\beta_i^H(t)\}_{i\in\mathbb{N}}$ is a sequence of stochastically independent one-dimensional fBm. Notice that the above series is convergent in $L^2(\Omega,\mathcal{F},\mathbb{P})$ since $\sum_{i=1}^\infty\lambda_i<\infty$ and $\mathbb{E}(\beta_i^H(t))^2=|t|^{2H}$ for $t\in\mathbb{R}$.

When $H=1/2$, $B^H(t)$ is the standard Brownian motion.

Throughout all the paper we assume that $H\in(1/2,1)$.

Using the definition of $B^H(t)$, Kolmogorov's theorem ensures that B^H has a continuous version. Thus we can consider the canonical interpretation of an fBm: let $\Omega=C_0(\mathbb{R},V)$, the space of continuous functions on $\mathbb{R}$ with values in V such that $\omega(0)=0$, equipped with the compact open topology. Let $\mathcal{F}$ be the associated Borel-σ-algebra and $\mathbb{P}$ the distribution of the fBm B^H, and $\{\theta_t\}_{t\in\mathbb{R}}$ be the flow of Wiener shifts, i.e.,

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\quad t\in\mathbb{R}.$$

Then the quadruple $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is a metric dynamical system which is ergodic, see [18,19,25]. Furthermore, it holds that

$$B^H(\cdot,\omega)=\omega(\cdot),\qquad B^H(\cdot,\theta_r\omega)=B^H(\cdot+r,\omega)-B^H(r,\omega)=\omega(\cdot+r)-\omega(r).\tag{2.2}$$

We also use the notation $\{\theta_t\}_{t\in\mathbb{R}}$ to shift the real fBm $\beta_i^H(t)\in\mathbb{R}$.

It is also known (see Kunita [20, Theorem 1.4.1]) that the fBm has a Hölder continuous version with Hölder exponent H' for each $H'\in(0,H)$ in the sense that there exists a set of probability one so that for every interval $[-N,N]$, $N\in\mathbb{N}$, there exists a constant $C=C_N(\omega)$ with

$$\|\omega\|_{C^{H'}(-N,N;V)}=\sup_{-N\leqslant s<t\leqslant N}\frac{\|\omega(t)-\omega(s)\|}{|t-s|^{H'}}\leqslant C.$$

It is easy to check that this set is $\{\theta_t\}_{t\in\mathbb{R}}$-invariant. For some fixed $1/2<H'<H$ the corresponding metric dynamical system has Hölder continuous paths θ-almost surely.

We now introduce the concept of random dynamical systems that is used to describe the dynamics of systems under the influence of a noise.

Definition 2.4. A random dynamical system (RDS) with one-sided time $\mathbb{T}^+$ and phase space V is a pair consisting of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and a mapping

$$\varphi : \mathbb{T}^+ \times \Omega \times V \to V$$

which is $(\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V))$-measurable and satisfies

$$\varphi(t, \theta_\tau \omega, \cdot) \circ \varphi(\tau, \omega, \cdot) = \varphi(t + \tau, \omega, \cdot) \quad \text{for } t, \tau \in \mathbb{T}^+,\ \omega \in \Omega \text{ (cocycle property)},$$
$$\varphi(0, \omega, \cdot) = \mathrm{id}_V.$$

φ is called a continuous RDS if the mapping

$$x \to \varphi(t, \omega, x)$$

is continuous for $t \in \mathbb{T}^+$, $\omega \in \Omega$.

A typical example is the solution operator of finite or infinite dimensional differential equations with random coefficients satisfying particular regularity assumptions. Another example is the solution operator of finite dimensional Ito-equations. For infinite dimensional Ito-equations with non-trivial diffusion coefficients this problem is rather unsolved. Recently, in [14] it has been shown that an infinite dimensional stochastic differential equation driven by an fBm with more general diffusion coefficients generates a random dynamical system.

The main purpose of this paper is to describe the dynamical behavior of an RDS given by an SPDE driven by an fBm. In particular we will describe the dynamical behavior in terms of random unstable manifolds.

Definition 2.5. Let $V = V^+ \oplus V^-$ be a splitting of V, where V^+ is finite dimensional. We consider an RDS φ such that $\varphi(t, \omega, 0) = 0$ for $t \in \mathbb{T}^+$, $\omega \in \Omega$. Let m be a mapping such that

- $m : \Omega \times V^+ \to V^-$ and $m(\omega, 0) = 0$.
- $m(\cdot, x^+) : \Omega \to V^-$ is measurable for every $x^+ \in V^+$.
- $m(\omega, \cdot) : V^+ \to V^-$ is Lipschitz for every $\omega \in \Omega$.

A set $M(\omega)$ defined by $\{x^+ + m(\omega, x^+):\ x^+ \in V^+\}$ is called a random unstable manifold at 0 if

(1) for every $x \in M(\omega)$, $t \in \mathbb{T}^+$, $\omega \in \Omega$ there exists an $x_{-t} \in M(\theta_{-t}\omega)$ such that $\varphi(t, \theta_{-t}\omega, x_{-t}) = x$ and

$$\lim_{t\to\infty} x_{-t} = 0, \quad \text{exponentially}, \tag{2.3}$$

(2) $\varphi(t, \omega, M(\omega)) \subset M(\theta_t\omega)$ for $t \in \mathbb{T}^+$, $\omega \in \Omega$.

For *local* manifolds we have

Definition 2.6. Let m be a mapping such that

- $m : \Omega \times U(\omega) \to V^-$, where $U(\omega) \subset V^+$ is a compact neighborhood of 0 and a random set with $m(\omega, 0) = 0$.

- m is measurable in the sense that

$$\omega \to m\big(\omega, x^+(\omega)\big)$$

  is measurable for every selector x^+ of U.
- $m(\omega,\cdot) : U(\omega) \to V^-$ is Lipschitz on $U(\omega)$.

Then $M(\omega) = \{x^+ + m(\omega, x^+)\colon\ x^+ \in U(\omega)\}$ is called *local* random unstable manifold at 0 if

(1) item (1) from Definition 2.5 holds for $x \in M(\omega) \cap W(\omega)$ where $W(\omega) \subset V$ is a random set and a neighborhood of 0,

(2)
$$\lim_{\|x\|\to 0} t_0(\omega, x) = \infty$$

where $x \in M(\omega)$ and

$$t_0(x,\omega) = \inf\big\{t \in \mathbb{T}^+\colon\ \varphi(t,\omega,x) \notin M(\theta_t\omega)\big\}.$$

We note that by the continuity of $m(\omega,\cdot)$ the set $M(\omega)$ is a random set. On the other hand, if φ is defined on $\mathbb{T}$ then (2.3) can be replaced by

$$\lim_{t\to-\infty} \varphi(t,\omega,x) = 0, \quad \varphi(t,\omega,x) \in M(\theta_t\omega),$$

for every $x \in M(\omega) \cap W(\omega)$ with exponential speed. We point out that a large class of random dynamical systems may be converted into the case that $x = 0$ is an equilibrium. See [1, p. 310].

3. Stochastic integrals and SPDE driven by an fBm

In this section we first introduce some basic concepts and results on fractional calculus and stochastic integrals with respect to the fBm β^H and B^H.

For $T > 0$, let $W^{\alpha,1}(0,T;V)$ be the space of measurable functions $f : [0,T] \to V$ such that

$$|f|_\alpha := \int_0^T \left(\frac{\|f(s)\|}{s^\alpha} + \int_0^s \frac{\|f(s) - f(\zeta)\|}{(s-\zeta)^{\alpha+1}}\, d\zeta \right) ds < \infty,$$

where $0 < \alpha < \frac{1}{2}$ is fixed and satisfies

$$1 - \alpha < H. \tag{3.1}$$

Following Zähle [36], for $f \in W^{\alpha,1}(0,T;V)$ we define the stochastic integral as the generalized Stieltjes integral

$$\int_0^T f\, d\beta^H = -\int_0^T D_{0+}^\alpha f(s) D_{T-}^{1-\alpha} \beta_{T-}^H(s)\, ds,$$

$$\int_s^t f\, d\beta^H = \int_0^T f \mathbf{1}_{(s,t)}\, d\beta^H \quad \text{for } s < t \in [0,T], \tag{3.2}$$

where, in general, for $0 < a < b < T$, $\beta^H_{b-}(s) := \beta^H(s) - \beta^H(b-)$ and $\beta^H(b-) := \lim_{\epsilon\to 0^+} \beta^H(b-\epsilon)$, and the Weyl derivatives are given by

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(t-a)^\alpha} + \alpha\int_a^t \frac{f(t)-f(\zeta)}{(t-\zeta)^{\alpha+1}}\,d\zeta\right),$$

$$D^{1-\alpha}_{b-} \beta^H(t) = \frac{1}{\Gamma(\alpha)}\left(\frac{\beta^H(t)}{(b-t)^{1-\alpha}} + (1-\alpha)\int_t^b \frac{\beta^H(t)-\beta^H(\zeta)}{(\zeta-t)^{2-\alpha}}\,d\zeta\right).$$

It can be proved (see, for instance, Nualart and Răşcanu [28], Decreusefond and Üstünel [10], Zähle [36]) that the stochastic integral (3.2) exists and that the following crucial inequality holds

$$\left\|\int_0^T f\,d\beta^H\right\| \leqslant \Lambda^{0,T}_\alpha\big(\beta^H\big)|f|_\alpha,$$

where

$$\Lambda^{0,T}_\alpha\big(\beta^H\big) := \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \sup_{0\leqslant s<t\leqslant T}\left(\frac{|\beta^H(s)-\beta^H(t)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|\beta^H(\zeta)-\beta^H(s)|}{(\zeta-s)^{2-\alpha}}\,d\zeta\right),$$

Γ is the Gamma function. Notice that $\Lambda^{0,T}_\alpha(\beta^H)$ is finite in a set of full measure which is $\{\theta_t\}_{t\in\mathbb{R}}$-invariant, which follows from the H'-Hölder continuity of the fBm for $H' \in (0,H)$ and (3.1).

Let us introduce

$$\Lambda^{0,T}_\alpha(\omega) := \sum_{i=1}^\infty \sqrt{\lambda_i}\,\Lambda^{0,T}_\alpha\big(\beta^H_i\big). \tag{3.3}$$

Lemma 3.1. *Suppose that $\sum_{i=1}^\infty \sqrt{\lambda_i} < \infty$. Then the random variable $\Lambda^{0,T}_\alpha(\omega)$, $T > 0$, is tempered from above.*

Proof. According to Proposition 4.1.3 in Arnold [1], it is sufficient to prove that

$$\mathbb{E} \sup_{r\in[0,1]} \Lambda^{0,T}_\alpha(\theta_r\omega) < \infty. \tag{3.4}$$

First of all, taking into account of the definition of the infinite dimensional fBm, we notice that $\theta_r\omega$ has the following expression

$$\theta_r\omega(t) = \sum_{i=1}^\infty \sqrt{\lambda_i}\,\theta_r\big(e_i\beta^H_i(t)\big)$$

for $t, r \in \mathbb{R}$. Thus,

$$\begin{aligned}
&\sup_{r\in[0,1]} \Lambda_\alpha^{0,T}(\theta_r\omega)\\
&\quad= \sup_{r\in[0,1]} \sum_{i=1}^\infty \sqrt{\lambda_i}\Lambda_\alpha^{0,T}\big(\theta_r\beta_i^H(t)\big)\\
&\quad= \sup_{r\in[0,1]} \sum_{i=1}^\infty \frac{\sqrt{\lambda_i}}{\Gamma(1-\alpha)\Gamma(\alpha)} \sup_{0\leqslant s<t\leqslant T}\left(\frac{|\theta_r\beta_i^H(s)-\theta_r\beta_i^H(t)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|\theta_r\beta_i^H(\zeta)-\theta_r\beta_i^H(s)|}{(\zeta-s)^{2-\alpha}}\,d\zeta\right)\\
&\quad\leqslant \sum_{i=1}^\infty \frac{\sqrt{\lambda_i}}{\Gamma(1-\alpha)\Gamma(\alpha)} \sup_{r\in[0,1]}\sup_{0\leqslant s<t\leqslant T}\left(\frac{|\theta_r\beta_i^H(s)-\theta_r\beta_i^H(t)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|\theta_r\beta_i^H(\zeta)-\theta_r\beta_i^H(s)|}{(\zeta-s)^{2-\alpha}}\,d\zeta\right)\\
&\quad\leqslant \sum_{i=1}^\infty \frac{\sqrt{\lambda_i}}{\Gamma(1-\alpha)\Gamma(\alpha)} \sup_{0\leqslant s<t\leqslant T+1}\left(\frac{|\beta_i^H(s)-\beta_i^H(t)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|\beta_i^H(\zeta)-\beta_i^H(s)|}{(\zeta-s)^{2-\alpha}}\,d\zeta\right).
\end{aligned}$$

We also have that

$$\begin{aligned}
&\mathbb{E}\sup_{0\leqslant s<t\leqslant T+1}\left(\frac{|\beta_i^H(s)-\beta_i^H(t)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|\beta_i^H(\zeta)-\beta_i^H(s)|}{(\zeta-s)^{2-\alpha}}\,d\zeta\right)\\
&\quad\leqslant \mathbb{E}\big\|\beta_i^H\big\|_{C^{H'}(0,T+1;\mathbb{R})}\sup_{0\leqslant s<t\leqslant T+1}\left((t-s)^{-1+\alpha+H'} + \int_s^t (\zeta-s)^{H'-(2-\alpha)}\,d\zeta\right)\\
&\quad\leqslant C\mathbb{E}\big\|\beta_i^H\big\|_{C^{H'}(0,T+1;\mathbb{R})} < \infty
\end{aligned}$$

where the last expression is independent of i, and finite due to the choice of α, see (3.1), and Kunita [20, Theorem 1.4.1]. Therefore, thanks to $\sum_{i=1}^\infty\sqrt{\lambda_i}<\infty$, (3.4) follows and then the proof is complete. □

Remark 3.2. The mapping $t\to\Lambda_\alpha^{0,T}(\theta_t\omega)$ is at most subexponentially increasing θ-almost surely. Hence we have the subexponential growth for every $\omega\in\Omega$ on a modified metric dynamical system.

Now we define the stochastic integral with respect to the infinite dimensional fBm B^H. Let $L(V)$ denote the space of linear bounded operators on V and let $G:\Omega\times[0,T]\to L(V)$ be an operator such that $G(\omega,\cdot)e_i\in W^{\alpha,1}(0,T;V)$ for each $i\in\mathbb{N}$ and $\omega\in\Omega$. We define

$$\int_0^T G\,d\omega := \sum_{i=1}^\infty \int_0^T G(s)Q^{1/2}e_i\,d\beta_i^H(s) = \sum_{i=1}^\infty\sqrt{\lambda_i}\int_0^T G(s)e_i\,d\beta_i^H(s), \tag{3.5}$$

where the convergence of the sums in (3.5) is understood in V.

The following result can be found in Masloswki and Nualart [24, Proposition 2.1].

Remark 3.3. Assume that $\sum_{i=1}^\infty\sqrt{\lambda_i}<\infty$. Then for all $\omega\in\Omega$ the pathwise integral (3.5) is well defined on Ω for each $G:\Omega\times[0,T]\to L(V)$ satisfying $G(\omega,\cdot)e_i\in W^{\alpha,1}(0,T;V)$, for $i\in\mathbb{Z}^+$ and $\omega\in\Omega$, such that $\sup_i|G(\omega,\cdot)e_i|_\alpha<\infty$, for $\omega\in\Omega$. In addition

$$\left\| \int_0^T G(s)\,d\omega(s) \right\| \leqslant \Lambda_\alpha^{0,T}(\omega) \sup_i \left| G(\cdot)e_i \right|_\alpha, \quad \omega \in \Omega, \tag{3.6}$$

where $\Lambda_\alpha^{0,T}(\omega)$ is given by (3.3).

Consider the following stochastic evolution equation in the separable Hilbert space V

$$\begin{cases} du(t) = \big(Au(t) + F\big(u(t)\big)\big)\,dt + G\big(u(t)\big)\,d\omega(t), \\ u(0) = u_0 \in V, \end{cases} \tag{3.7}$$

where ω denotes the infinite dimensional fBm B^H (see (2.2)) with covariance function Q such that $\sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty$.

Assume that A is the generator of an analytic semigroup $S(\cdot)$ such that the spectrum is discrete. In particular, all its eigenvalues are real and can be ordered as $\mu_1 > \mu_2 > \cdots > \mu_n > \mu_{n+1}, \ldots,$ $\lim_{n\to\infty} \mu_n = -\infty$. We assume that A is hyperbolic, i.e., 0 is not an eigenvalue. Let V^+ be the linear subspace of V spanned by the eigenvectors with eigenvalues larger than 0. This space is then finite dimensional. Let V^- be the linear subspace of V spanned by the eigenvectors with eigenvalues less than 0. Then we have the splitting

$$V = V^+ \oplus V^-.$$

Let us denote by $\pi^\pm$ the orthogonal projections associated with this splitting. We consider the restriction of the semigroup S to V^+, V^- denoted by $S^\pm$. Clearly, $\pi^\pm$ commutes with $S(t)$ and

$$S^\pm(t) : V^\pm \to V^\pm \quad \text{for } t \geqslant 0.$$

Since S^+ is generated by the bounded operator $\pi^+ A \pi^+$ on V^+, we can extend S^+ to $\mathbb{R}$. Since A is hyperbolic and generates an analytic semigroup, there exist positive constants $\hat{\mu}$ and C_S and a negative constant $\check{\mu}$ such that

$$\left\| S^+(t) \right\| \leqslant C_S e^{\hat{\mu} t} \quad \text{for } t \leqslant 0, \qquad \left\| S^-(t) \right\| \leqslant C_S e^{\check{\mu} t} \quad \text{for } t \geqslant 0,$$

where $\|\cdot\|$ also denotes the norm of $L(V^\pm)$. We mention that $\hat{\mu}$ can be any positive number less than the smallest positive eigenvalue of A and $\check{\mu}$ can be any negative number larger than the largest negative eigenvalue of A.

Notice that the operator $A\pi^-$ is strictly negative definite on V^- (although A is not strictly negative) and that $\pi^+ A$ is a finite dimensional operator. Let V_δ, $\delta \geqslant 0$, denote the domain of $\pi^+ \,\mathrm{id}_V + \pi^-(-A\pi^-)^\delta$ equipped with the norm $\|x\|_{V_\delta} := \|\pi^+ x\| + \|\pi^-(-A\pi^-)^\delta x\|$, $x \in V_\delta$. When $\delta = 0$ we simply write $\|x\|_{V_0} = \|x\|$.

We recall here some properties of the analytic semigroup, which will be used later in our analysis. For $0 \leqslant s < t$,

$$\left\| S(t-s) \right\|_{L(V_\gamma, V_\beta)} \leqslant C(t-s)^{-\beta+\gamma} \quad \text{for } 0 \leqslant \gamma < \beta \leqslant 1, \tag{3.8}$$

$$\left\| S(t-s) - \mathrm{id} \right\|_{L(V_{\gamma+\mu}, V_\gamma)} \leqslant C(t-s)^{\mu} \quad \text{for } \gamma \in [0,1),\ \mu \in (0, 1-\gamma) \tag{3.9}$$

(see, for instance, [29]).

We assume that F is Lipschitz continuous with Lipschitz constant L_F, and G and $G' : V \to L(V, L(V))$ are Lipschitz continuous in the following senses:

$$\sup_{i\in\mathbb{N}}\left\|G(v_1)e_i-G(v_2)e_i\right\|\leqslant L_G\|v_1-v_2\|, \tag{3.10}$$

$$\sup_{i\in\mathbb{N}}\left\|G'(v_1)e_i-G'(v_2)e_i\right\|_{L(V)}\leqslant L'_G\|v_1-v_2\|, \tag{3.11}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is the complete orthonormal basis in V introduced in Section 2.

By the solution of (3.7) on $[0,T]$ we mean a V-valued process u whose paths are for every $T>0$ and $\omega\in\Omega$ elements of $W^{\alpha,1}(0,T;V)$, for an $\alpha\in(1-H,\frac{1}{2})$, and such that

$$u(t)=S(t)u_0+\int_0^t S(t-s)F\big(u(s)\big)\,ds+\int_0^t S(t-s)G\big(u(s)\big)\,d\omega,\quad t\in[0,T], \tag{3.12}$$

where the stochastic integral has to be understood according to (3.5).

For such an $\alpha\in(1-H,\frac{1}{2})$, denote by $W_\xi^{\alpha,\infty}(0,T;V)$ the Banach space of measurable functions $x:[0,T]\to V$ such that $\|x\|_{\alpha,\xi}<\infty$, where

$$\|x\|_{\alpha,\xi}:=\sup_{t\in[0,T]}\left(\left\|x(t)\right\|+t^\xi\int_0^t\frac{\|x(t)-x(r)\|}{(t-r)^{1+\alpha}}\,dr\right)$$

for $\xi\in[\alpha,1-\alpha)$.

Let us also introduce the Banach space $W^{\alpha,\delta,\infty}(0,T;V)$ of measurable functions $x:[0,T]\to V$ such that

$$|x|_{\alpha,\delta}:=\sup_{t\in[0,1]}\left(\left\|x(t)\right\|_{V_\delta}+\int_0^t\frac{\|x(t)-x(r)\|}{(t-r)^{1+\alpha}}\,dr\right)<\infty$$

for $\alpha\in(1-H,\frac{1}{2})$, $\delta\in[0,1)$.

Remark 3.4. Note that $W^{\alpha,\delta,\infty}(0,T;V)$ is continuously embedded in $W_\xi^{\alpha,\infty}(0,T;V)$, and the latter is continuously embedded in $W^{\alpha,1}(0,T;V)$.

We set

$$\mathcal{L}(v):=\int_0^{\cdot} S(\cdot-\tau)v(\tau)\,d\omega(\tau),$$

for $v\in W_{\xi,\mathcal{L}}^{\alpha,\infty}$ where

$$W_{\xi,\mathcal{L}}^{\alpha,\infty}:=\left\{v(t)\in L(V):\ v(\cdot)e_i\in W_\xi^{\alpha,\infty}(0,T;V)\text{ for each }i\in\mathbb{N},\ \text{with }\sup_{i\in\mathbb{N}}\|v(\cdot)e_i\|_{\alpha,\xi}<\infty\right\}.$$

Then, it can be proven, see [14], that if $v\in W_{\xi,\mathcal{L}}^{\alpha,\infty}$ then $\mathcal{L}(v)$ is well defined.

We also introduce, for $v:[0,T]\to V$ measurable with $\sup_{t\in[0,T]}\|v(t)\|<\infty$,

$$\mathcal{I}(v):=\int_0^{\cdot} S(\cdot-\tau)v(\tau)\,d\tau.$$

The proof of the next result can be found in [14].

Lemma 3.5. *Let $\alpha \in (1-H, \frac{1}{2})$ and $\xi \in [\alpha, 1-\alpha)$. Then the following statements hold*:

(i) *For $v \in W^{\alpha,\infty}_{\xi,\mathcal{L}}$ we have*

$$\left\|\mathcal{L}(v)\right\|_{\alpha,\xi} \leqslant C_1\left(\Lambda^{0,T}_{\alpha}(\omega)\right)\sup_{i\in\mathbb{N}}\left\|v(\cdot)e_i\right\|_{\alpha,\xi}, \tag{3.13}$$

where C_1 is a tempered from above random variable. In addition, if $\sup_{i\in\mathbb{N}}|v(\cdot)e_i|_{\alpha,0} < \infty$, then for $\delta \in [0, 1-\alpha)$,

$$\left|\mathcal{L}(v)\right|_{\alpha,\delta} \leqslant C_2\left(\Lambda^{0,T}_{\alpha}(\omega)\right)\sup_{i\in\mathbb{N}}\left|v(\cdot)e_i\right|_{\alpha,0}, \tag{3.14}$$

where $C_2(\Lambda^{0,T}_{\alpha}(\omega))$ is also a tempered from above random variable.

(ii) *For $v : [0,T] \to V$ measurable such that $\sup_{t\in[0,T]}\|v(t)\| < \infty$, we have*

$$\left\|\mathcal{I}(v)\right\|_{\alpha,\xi} \leqslant C_3 \sup_{t\in[0,T]}\left\|v(t)\right\|, \tag{3.15}$$

where C_3 is a positive constant. In addition, there is another positive constant C_4 such that for $\delta \in [0,1)$

$$\left|\mathcal{I}(v)\right|_{\alpha,\delta} \leqslant C_4 \sup_{t\in[0,T]}\left\|v(t)\right\|. \tag{3.16}$$

We consider the operators $\mathcal{G}$ and $\mathcal{F}$ associated respectively to G and F, which are given by

$$\mathcal{G}(u)(t) := G\big(u(t)\big), \qquad \mathcal{F}(u)(t) := F\big(u(t)\big), \quad t \in [0,T].$$

Then, from now on we write Eq. (3.12) in the equivalent shorter way

$$u(t) = S(t)u_0 + \mathcal{D}(\omega, u)(t), \quad t \in [0,T], \tag{3.17}$$

with $u_0 \in V$ and

$$\mathcal{D}(\omega,u) := \mathcal{I}\circ\mathcal{F}(u) + \mathcal{L}\circ\mathcal{G}(u).$$

Let us check that Eq. (3.17) is well defined on $W^{\alpha,\infty}_{\xi}(0,T;V)$. Assume that $u \in W^{\alpha,\infty}_{\xi}(0,T;V)$. On one hand, because of (3.10), we have

$$\begin{aligned}\sup_i\left\|\mathcal{G}(u)(\cdot)e_i\right\|_{\alpha,\xi} &\leqslant \sup_i \sup_{t\in[0,T]}\Bigg(\left\|G\big(u(t)\big)e_i - G(0)e_i\right\| + \left\|G(0)e_i\right\| \\ &\quad + t^{\xi}\int_0^t \frac{\|G(u(t))e_i - G(u(s))e_i\|}{(t-s)^{1+\alpha}}\,ds\Bigg) \\ &\leqslant \sup_i\left\|G(0)e_i\right\| + L_G\|u\|_{\alpha,\xi},\end{aligned} \tag{3.18}$$

thus $\mathcal{G}(u) \in W^{\alpha,\infty}_{\xi,\mathcal{L}}$. On the other hand, thanks to the Lipschitz continuity of F,

$$\sup_{t\in[0,T]} \big\|\mathcal{F}(u)(t)\big\| \leqslant \sup_{t\in[0,T]} \big\|F\big(u(t)\big) - F(0)\big\| + \big\|F(0)\big\| \leqslant \big\|F(0)\big\| + L_F \sup_{t\in[0,T]} \big\|u(t)\big\|$$
$$\leqslant \big\|F(0)\big\| + L_F \|u\|_{\alpha,\xi}. \tag{3.19}$$

Therefore, (3.18) and (3.19), together with Lemma 3.5, ensure that $\mathcal{D}$ is well defined on $W^{\alpha,\infty}_{\xi}(0,T;V)$.

In addition,

$$\begin{aligned}\big\|S(\cdot)u_0\big\|_{\alpha,\xi} &= \sup_{t\in[0,T]}\left(\big\|S(t)u_0\big\| + t^{\xi}\int_0^t \frac{\|(S(t)-S(s))u_0\|}{(t-s)^{1+\alpha}}\,ds\right)\\
&\leqslant \sup_{t\in[0,T]}\left(C\|u_0\| + t^{\xi}\int_0^t \frac{\|(S(t-s)-\mathrm{id})S(s)u_0\|}{(t-s)^{1+\alpha}}\,ds\right)\\
&\leqslant \sup_{t\in[0,T]}\left(C\|u_0\| + t^{\xi}\int_0^t \frac{(t-s)^{\beta}\|S(s)u_0\|_{V_\beta}}{(t-s)^{1+\alpha}}\,ds\right)\\
&\leqslant \|u_0\| \sup_{t\in[0,T]}\big(C + t^{\xi-\alpha}B(1-\beta,\beta-\alpha)\big) \leqslant C\|u_0\| < \infty,\end{aligned} \tag{3.20}$$

where β is a constant in the interval $(\alpha,1]$, and $B(1-\beta,\beta-\alpha)$ is the Beta function.

In order to ensure that (3.20) is finite, notice that it has been essential to have the term t^{ξ} in the definition of $\|\cdot\|_{\alpha,\xi}$, since $\sup_{t\in[0,T]} t^{-\alpha}B(1-\beta,\beta-\alpha) = \infty$.

The following theorem on the existence and uniqueness of solutions to Eq. (3.17) has been proved in [14].

Theorem 3.6. *Let $\alpha \in (1-H, \frac{1}{2})$ and $\xi \in [\alpha, 1-\alpha)$. Assume F is Lipschitz continuous, and that G and G' satisfy* (3.10) *and* (3.11) *respectively. Then, for each initial point $u_0 \in V$ there exists a unique solution to Eq.* (3.17) *with its paths in $W^{\alpha,\infty}_{\xi}(0,T;V)$. In addition, the mapping $\Phi : V \to W^{\alpha,\infty}_{\xi}(0,T;V)$ given by $\Phi : u_0 \mapsto u$ is continuous for $\omega \in \Omega$.*

The next theorem summarizes some properties about the solution of (3.17). In particular, the stochastic equation generates a random dynamical system φ in such a way that $\varphi(t,\omega,u_0)$ is defined by the solution of the stochastic evolution equation at time t, for a noise path ω, with initial point u_0.

Theorem 3.7. *The solution $u(t,\omega,u_0)$ of Eq.* (3.17) *satisfies the following*:

(a) *$u(t) \in V_\delta$, for $\delta \in [0, 1-\xi)$, $\xi \in [\alpha, 1-\alpha)$ and $t \in (0,T]$.*
(b) *The solution u of* (3.17) *defines a random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$, given by*

$$\varphi(t,\omega,u_0) = S(t)u_0 + \int_0^t S(t-s)F\big(u(s)\big)\,ds + \int_0^t S(t-s)G\big(u(s)\big)\,d\omega.$$

The proof of this theorem can be also found in [14].

4. Fixed points of the Lyapunov–Perron transform

In this section, we introduce the *Lyapunov–Perron transform* associated with the RDS generated by Eq. (3.17) over a space of infinite sequences whose terms are functions in $W^{\alpha,\delta,\infty}(0,T;V)$ for $\delta > 1/2$, $\alpha+\delta<1$. The fixed point of this transform gives a pre-form of the random unstable manifold we intent to obtain.

From now on, we will work in the space $W^{\alpha,\delta,\infty}(0,T;V)$ instead of $W_\xi^{\alpha,\infty}(0,T;V)$ since we will obtain unstable invariant manifolds in the space V_δ (see Section 5). Note that, as an immediate consequence of Theorem 3.7, we could consider smooth initial conditions in the space V_δ instead of in V, for $\delta \in [0,1-\xi)$. In addition, in the particular case that $u_0 \in V_\delta$, with $\delta \geqslant \alpha$, we can prove the existence and uniqueness of solutions to (3.17) in the space $W^{\alpha,\delta,\infty}(0,T;V)$ instead of $W_\xi^{\alpha,\infty}(0,T;V)$, since then

$$\begin{aligned}\left|S(\cdot)u_0\right|_{\alpha,\delta} &= \sup_{t\in[0,T]}\left(\left\|S(t)u_0\right\|_{V_\delta} + \int_0^t \frac{\|(S(t)-S(s))u_0\|}{(t-s)^{1+\alpha}}\,ds\right) \\ &\leqslant \sup_{t\in[0,T]}\left(C\|u_0\|_{V_\delta} + \int_0^t \frac{\|(S(t-s)-\mathrm{id})S(s)u_0\|}{(t-s)^{1+\alpha}}\,ds\right) \\ &\leqslant \|u_0\|_{V_\delta}\sup_{t\in[0,T]}\big(C+t^{\delta-\alpha}CB(1+\delta-\beta,\beta-\alpha)\big) = C_\delta\|u_0\|_{V_\delta},\end{aligned} \tag{4.1}$$

where $\beta \in (\alpha,1]$.

The finite dimensional version of (4.1) is given by

$$\left|S^+(\cdot)u_0\right|_{\alpha,\delta} \leqslant C^+\|u_0\|, \quad u_0 \in V^+. \tag{4.2}$$

From now on, we are moreover working on time interval $[0,1]$, i.e. $T=1$.

In order to prove the existence of a fixed point of the Lyapunov–Perron transform we need to modify the operator $\mathcal{D}(\omega,\cdot)$ so that the Lipschitz constants are small. For this we modify the mappings $\mathcal{F}$ and $\mathcal{G}$ by using a cut-off function.

Let $\chi : W^{\alpha,\delta,\infty}(0,1;V) \to W^{\alpha,\delta,\infty}(0,1;V)$ be a Lipschitz continuous cut-off function:

$$\chi(u) = \begin{cases} u, & |u|_{\alpha,\delta} \leqslant \frac{1}{2}, \\ 0, & |u|_{\alpha,\delta} \geqslant 1, \end{cases}$$

with Lipschitz constant denoted by L_χ. For instance, taking $q:\mathbb{R}^+ \to [0,1]$

$$q(x) = \begin{cases} 1, & x \leqslant \frac{1}{2}, \\ 2-2x, & x \in (\frac{1}{2},1), \\ 0, & x \geqslant 1, \end{cases}$$

we can set $\chi(u) = uq(|u|_{\alpha,\delta})$ which has a Lipschitz constant $L_\chi = 3$ and $|\chi(u)|_{\alpha,\delta} \leqslant 1$. In the following we assume that χ is constructed using q.

For a positive number R, we define

$$\chi_R(u) = R\chi\left(\frac{u}{R}\right). \tag{4.3}$$

We assume

$$F(0)=0, \qquad G(0)=0, \qquad G'(0)=0 \tag{4.4}$$

so that $u=0$ is a stationary solution of the equation we study. We introduce the operators

$$\mathcal{G}_R(u):=\mathcal{G}\circ\chi_R(u), \qquad \mathcal{F}_R(u):=\mathcal{F}\circ\chi_R(u), \quad u\in W^{\alpha,\delta,\infty}(0,1;V).$$

It is straightforward to see that if $|u|_{\alpha,\delta}\leqslant\frac{R}{2}$, then $\mathcal{G}_R(u)=\mathcal{G}(u)$ and $\mathcal{F}_R(u)=\mathcal{F}(u)$. Let $L_{\mathcal{G}}(R)$ and $L_{\mathcal{F}}(R)$ denote the Lipschitz constants of $\mathcal{G}_R$ and $\mathcal{F}_R$, respectively, which are supposed to be strictly increasing in R.

Lemma 4.1. *Suppose that G satisfies the assumptions* (3.10), (3.11) *and* (4.4). *Then*

$$\sup_{i\in\mathbb{N}}\bigl|\mathcal{G}_R(u_1)(\cdot)e_i-\mathcal{G}_R(u_2)(\cdot)e_i\bigr|_{\alpha,0}\leqslant L_{\mathcal{G}}(R)|u_1-u_2|_{\alpha,\delta}, \qquad L_{\mathcal{G}}(R)=c_\delta^2L'_G(3L_\chi+1)R,$$

where c_δ denotes the embedding constant of $W^{\alpha,\delta,\infty}(0,1;V)\subset W^{\alpha,0,\infty}(0,1;V)$.

Proof. By definition, we have

$$\begin{aligned}\bigl|\mathcal{G}\bigl(\chi_R(u_1)\bigr)(\cdot)e_i-\mathcal{G}\bigl(\chi_R(u_2)\bigr)(\cdot)e_i\bigr|_{\alpha,0}&=\sup_{t\in[0,1]}\Biggl(\bigl\|G\bigl(\chi_R(u_1)(t)\bigr)e_i-G\bigl(\chi_R(u_2)(t)\bigr)e_i\bigr\|\\&\quad+\int_0^t\frac{\|G(\chi_R(u_1)(t))e_i-G(\chi_R(u_2)(t))e_i-(G(\chi_R(u_1)(\lambda))e_i-G(\chi_R(u_2)(\lambda)))e_i\|}{(t-\lambda)^{1+\alpha}}\,d\lambda\Biggr).\end{aligned}$$

Using the Lipschitz continuity of G', we obtain

$$\begin{aligned}&\bigl\|G\bigl(\chi_R(u_1)(t)\bigr)e_i-G\bigl(\chi_R(u_2)(t)\bigr)e_i\bigr\|\\&\quad\leqslant L'_G\max\bigl(\bigl\|\chi_R(u_1)(t)\bigr\|,\bigl\|\chi_R(u_2)(t)\bigr\|\bigr)\bigl\|\chi_R(u_1)(t)-\chi_R(u_2)(t)\bigr\|\\&\quad\leqslant L'_Gc_\delta\max\bigl(\bigl|\chi_R(u_1)\bigr|_{\alpha,\delta},\bigl|\chi_R(u_2)\bigr|_{\alpha,\delta}\bigr)\bigl\|\chi_R(u_1)(t)-\chi_R(u_2)(t)\bigr\|\\&\quad\leqslant L'_Gc_\delta R\bigl\|\chi_R(u_1)(t)-\chi_R(u_2)(t)\bigr\|.\end{aligned}$$

Thus, applying Lemma 7.1 in [28], we get

$$\begin{aligned}&\bigl\|G\bigl(\chi_R(u_1)(t)\bigr)e_i-G\bigl(\chi_R(u_2)(t)\bigr)e_i-\bigl(G\bigl(\chi_R(u_1)(\lambda)\bigr)e_i-G\bigl(\chi_R(u_2)(\lambda)\bigr)e_i\bigr)\bigr\|\\&\quad\leqslant L'_Gc_\delta R\bigl\|\chi_R(u_1)(t)-\chi_R(u_2)(t)-\bigl(\chi_R(u_1)(\lambda)-\chi_R(u_2)(\lambda)\bigr)\bigr\|\\&\qquad+L'_G\bigl\|\chi_R(u_1)(t)-\chi_R(u_2)(t)\bigr\|\bigl(\bigl\|\chi_R(u_1)(t)-\chi_R(u_1)(\lambda)\bigr\|+\bigl\|\chi_R(u_2)(t)-\chi_R(u_2)(\lambda)\bigr\|\bigr).\end{aligned}$$

Indeed, we have $\|\chi_R(u)(t)\|\leqslant|\chi_R(u)|_{\alpha,0}\leqslant c_\delta|\chi_R(u)|_{\alpha,\delta}\leqslant c_\delta R$ for $t\in[0,1]$. Hence the Lipschitz constant M_0 in Lemma 7.1 in [28] can be chosen as $L'_Gc_\delta R$. Thus

$$\begin{aligned}&\big|\mathcal{G}\big(\chi_R(u_1)\big)(\cdot)e_i-\mathcal{G}\big(\chi_R(u_2)\big)(\cdot)e_i\big|_{\alpha,0}\\&\quad\leqslant L'_G c_\delta^2 R\big|\chi_R(u_1)-\chi_R(u_2)\big|_{\alpha,\delta}+c_\delta^2 L'_G\big|\chi_R(u_1)-\chi_R(u_2)\big|_{\alpha,\delta}\big(\big|\chi_R(u_1)\big|_{\alpha,\delta}+\big|\chi_R(u_2)\big|_{\alpha,\delta}\big)\\&\quad\leqslant 3c_\delta^2 R L'_G L_\chi|u_1-u_2|_{\alpha,\delta}. \qquad\square\end{aligned}$$

Remark 4.2. If F fulfills the similar properties as G (that is, F differentiable such that $F'(0)=0$), then one can prove a similar statement given by Lemma 4.1 for $\mathcal{F}_R$. However, we here are assuming a weaker condition for F than for G (see Section 6).

Assume for $\mathcal{F}_R$ that

$$\sup_{t\in[0,1]}\big\|\mathcal{F}_R(u_1)(t)-\mathcal{F}_R(u_2)(t)\big\|\leqslant L_{\mathcal{F}}(R)|u_1-u_2|_{\alpha,\delta},\quad u_1,u_2\in W^{\alpha,\delta,\infty}(0,1;V). \tag{4.5}$$

An example for $\mathcal{F}$ satisfying such a condition is given in Section 6.

For a random variable $R(\omega)>0$, we set

$$\mathcal{D}_{R(\omega)}(\omega,\cdot):=\mathcal{I}\circ\mathcal{F}_{R(\omega)}+\mathcal{L}\circ\mathcal{G}_{R(\omega)}:W^{\alpha,\delta,\infty}(0,1;V)\to W^{\alpha,\delta,\infty}(0,1;V)\quad\text{for }\omega\in\Omega.$$

We abbreviate $\mathcal{D}_{R(\omega)}(\omega,u)$ by $\mathcal{D}_R(\omega,u)$ and $\mathcal{F}_{R(\omega)}(u)=\mathcal{F}_R(\omega,u)$, $\mathcal{G}_{R(\omega)}(u)=\mathcal{G}_R(\omega,u)$.

Let $C_{\mathcal{L}}(\omega)$ denote the random variable appearing in (3.14) and $C_{\mathcal{I}}$ denote the constant in (3.16). For $K>0$ let $\tilde{R}(\omega)$ be the unique solution of

$$C_{\mathcal{L}}(\omega)L_{\mathcal{G}}\big(\tilde{R}(\omega)\big)+C_{\mathcal{I}}L_{\mathcal{F}}\big(\tilde{R}(\omega)\big)=K, \tag{4.6}$$

and let $R(\omega):=\min(\tilde{R}(\omega),1)$. Note that $C_{\mathcal{L}}$ is tempered from above. Because of the form of $L_{\mathcal{G}}(R)$ (see Lemma 4.1), assuming that $L_{\mathcal{F}}(R)$ is increasing and continuous (see Lemma 6.3 below for a particular example), we obtain that R is a tempered from below random variable.

We also have for every $\omega\in\Omega$ and $u_1,u_2\in W^{\alpha,\delta,\infty}(0,1;V)$ that

$$\big|\mathcal{D}_R(\omega,u_1)-\mathcal{D}_R(\omega,u_2)\big|_{\alpha,\delta}\leqslant K|u_1-u_2|_{\alpha,\delta}, \tag{4.7}$$

where $0\leqslant\delta<1-\alpha$.

For the following we choose a δ so that $\alpha\leqslant\delta<1-\alpha$ (this restriction comes from (4.1)).

Define

$$\kappa=\frac{\hat{\mu}+\check{\mu}}{2},\qquad \mu=\frac{\hat{\mu}-\check{\mu}}{2}>0.$$

Let $\mathcal{H}_\kappa$ be the space of sequences $U=(u_{i-1})_{i\in\mathbb{Z}^-}$, $u_i\in W^{\alpha,\delta,\infty}(0,1;V)$, with finite norm

$$\|U\|_{\mathcal{H}_\kappa}:=\sup_{i\in\mathbb{Z}^-}e^{-\kappa(i-1)}|u_{i-1}|_{\alpha,\delta}\quad\text{and}\quad u_{i-1}(0)=u_{i-2}(1),\quad i\in\mathbb{Z}^-.$$

We introduce the notation $U(i-1,t)=u_{i-1}(t)$, $t\in[0,1]$ and $U(\tau)=U(i,t)$ if $\tau=t+i-1$. Let also $\mathcal{H}_\kappa^{u_0^+}$ be the complete subset of $\mathcal{H}_\kappa$ such hat $\pi^+U(-1,1)=u_0^+\in V^+$.

The following mapping J on $\Omega\times\mathcal{H}_\kappa^{u_0^+}$

$$
\begin{aligned}
J(\omega,U)(i-1,t)=&\sum_{k=-\infty}^{i-1}S^-(t+i-1-k)\mathcal{D}_R^-(\theta_{k-1}\omega,u_{k-1})(1)+\mathcal{D}_R^-(\theta_{i-1}\omega,u_{i-1})(t)\\
&-\sum_{i+1}^{k=0}S^+(t+i-1-k)\mathcal{D}_R^+(\theta_{k-1}\omega,u_{k-1})(1)-\hat{\mathcal{D}}_R^+(\theta_{i-1}\omega,u_{i-1})(t)\\
&+S^+(t+i-1)u_0^+,\quad t\in[0,1],\ \omega\in\Omega,\ i\in\mathbb{Z}^-,
\end{aligned}
\tag{4.8}
$$

where $\mathcal{D}_R^-=\pi^-\mathcal{D}_R$, $\mathcal{D}_R^+=\pi^+\mathcal{D}_R$, and

$$
\hat{\mathcal{D}}_R^+(\omega,u)(t)=\int_t^1 S^+(t-\tau)\mathcal{G}_R(u)(\tau)\,d\omega(\tau)+\int_t^1 S^+(t-\tau)\mathcal{F}_R(u)(\tau)\,d\tau,
$$

is called *Lyapunov–Perron transform*.[2]

Remark 4.3. The reason to call this mapping Lyapunov–Perron transform is that under the condition $|U(i-1,\cdot)|_{\alpha,\delta}\leqslant R(\theta_{i-1}\omega)/2$, $i\in\mathbb{Z}^-$ we can write this mapping as

$$
\begin{aligned}
J(\omega,U)(i-1,t)=&\int_{-\infty}^{s}S^-(s-\tau)G\big(U(\tau)\big)\,d\omega(\tau)+\int_{-\infty}^{s}S^-(s-\tau)F\big(U(\tau)\big)\,d\tau\\
&+S^+(s)u_0^++\int_0^s S^+(s-\tau)G\big(U(\tau)\big)\,d\omega(\tau)\\
&+\int_0^s S^+(s-\tau)F\big(U(\tau)\big)\,d\tau,
\end{aligned}
$$

$s=i-1+t\leqslant 0$, which is the usual Lyapunov–Perron transform for continuous time, see for instance Lu and Schmalfuß [21].

Remark 4.4. It can be checked that

$$
\left|\int_t^1 S^+(t-\tau)v_1(\tau)\,d\omega(\tau)-\int_t^1 S^+(t-\tau)v_2(\tau)\,d\omega(\tau)\right|_{\alpha,\delta}\leqslant C\big(\Lambda_\alpha^{0,1}(\omega)\big)|v_1-v_2|_{\alpha,\delta},
$$

and also

$$
\left|\int_t^1 S^+(t-\tau)v_1(\tau)\,d\tau-\int_t^1 S^+(t-\tau)v_2(\tau)\,d\tau\right|_{\alpha,\delta}\leqslant C\sup_{t\in[0,1]}\big\|v_1(t)-v_2(t)\big\|.
$$

[2] The summation for $\sum_i^{k=j}$ is given starting with $k=j$ in negative direction. If $i>j$ this sum is zero.

In particular, these estimates together with Lemma 4.1 and assumption (4.5) imply

$$\left|\hat{\mathcal{D}}_R^+(\omega,u_1)-\hat{\mathcal{D}}_R^+(\omega,u_2)\right|_{\alpha,\delta}\leqslant K|u_1-u_2|_{\alpha,\delta}$$

for every $\omega\in\Omega$ and $u_1,u_2\in W^{\alpha,\delta,\infty}(0,1;V)$.

Theorem 4.5. *Let the constant K from* (4.6) *be defined by*

$$K^{-1}=4e^{\mu}\left(\frac{e^{-\kappa}C_S(C_\delta+C^+)+1}{1-e^{-\mu}}\right)$$

where C_δ and C^+ are given by (4.1) *and* (4.2) *respectively. Then, we have the following*:

(i) *For each $\omega\in\Omega$, the mapping $J(\omega,\cdot)$ maps $\mathcal{H}_\kappa^{u_0^+}$ into $\mathcal{H}_\kappa^{u_0^+}$.*
(ii) *For every $u_0^+\in V^+$ and $\omega\in\Omega$, $J(\omega,\cdot)$ has a unique fixed point $\Gamma(u_0^+,\omega)\in\mathcal{H}_\kappa^{u_0^+}$.*
(iii) *The mapping*

$$V^+\ni u_0^+\to\Gamma\left(u_0^+,\omega\right)\in\mathcal{H}_\kappa^{u_0^+}$$

is Lipschitz continuous with Lipschitz constant L_Γ.

Proof. (i) We estimate

$$\begin{aligned}
e^{-\kappa(i-1)}\left|S^+(\cdot)S^+(i-1)u_0^+\right|_{\alpha,\delta}\leqslant e^{-\kappa(i-1)}\sup_{t\in[0,1]}\Bigg(&\left\|S^+(i-1)S^+(t)u_0^+\right\|\\
&+\int_0^t\frac{\|S^+(i-1)S^+(t)u_0^+-S^+(i-1)S^+(\lambda)u_0^+\|}{(t-\lambda)^{1+\alpha}}\,d\lambda\Bigg)\\
\leqslant C_Se^{(\hat{\mu}-\kappa)(i-1)}\left|S^+(\cdot)u_0^+\right|_{\alpha,\delta}&\leqslant C_Se^{\mu(i-1)}C^+\left\|u_0^+\right\|,
\end{aligned}\tag{4.9}$$

which is bounded with respect to $i\in\mathbb{Z}^-$.

The relationship

$$J(\omega,U)(i-1,0)=J(\omega,U)(i-2,1)$$

follows from

$$\begin{aligned}
&\int_t^1 S^+(t-\tau)\mathcal{F}_R(\omega,u_{i-1})(\tau)\,d\tau+\int_t^1 S^+(t-\tau)\mathcal{G}_R(\omega,u_{i-1})(\tau)\,d\theta_{i-1}\omega\Big|_{t=0}\\
&\quad=S^+(-1)\mathcal{D}_R^+(\theta_{i-1}\omega,u_{i-1})(1).
\end{aligned}$$

We note that $J(\omega,U)\in\mathcal{H}_\kappa$ follows from (ii) below with $U^2=0$, the fact that $\mathcal{F}_R(0)=0$, $\mathcal{G}_R(0)=0$, and (4.9). Moreover, it is straightforward to check that $\pi^+J(\omega,U)(-1,1)=u_0^+$, so that $J(\omega,U)\in\mathcal{H}_\kappa^{u_0^+}$.

(ii) We prove that the mapping J is a contraction. Let $U^j=(u^j_{i-1})_{i\in\mathbb{Z}^-}\in\mathcal{H}_\kappa^{u_0^+}$ for $j=1,2$. For every $i\in\mathbb{Z}^-$ we have

$$\begin{aligned}
&\sum_{k=-\infty}^{i-1}\big|S^-(\cdot+i-1-k)\big(\mathcal{D}_R^-\big(\theta_{k-1}\omega,u^1_{k-1}\big)(1)-\mathcal{D}_R^-\big(\theta_{k-1}\omega,u^2_{k-1}\big)(1)\big)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\qquad+\big|\mathcal{D}_R^-\big(\theta_{i-1}\omega,u^1_{i-1}\big)(\cdot)-\mathcal{D}_R^-\big(\theta_{i-1}\omega,u^2_{i-1}\big)(\cdot)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\quad\leqslant\sum_{k=-\infty}^{i-1}C_Se^{(\check\mu-\kappa)(i-k-1)}e^{-\kappa(k-1)}e^{-\kappa}C_\delta\big\|\mathcal{D}_R^-\big(\theta_{k-1}\omega,u^1_{k-1}\big)(1)-\mathcal{D}_R^-\big(\theta_{k-1}\omega,u^2_{k-1}\big)(1)\big\|_{V_\delta}\\
&\qquad+\big|\mathcal{D}_R^-\big(\theta_{i-1}\omega,u^1_{i-1}\big)(\cdot)-\mathcal{D}_R^-\big(\theta_{i-1}\omega,u^2_{i-1}\big)(\cdot)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\quad\leqslant C_S\sum_{k=-\infty}^{i-1}e^{(\check\mu-\kappa)(i-k-1)}e^{-\kappa(k-1)}Ke^{-\kappa}C_\delta\big|u^1_{k-1}-u^2_{k-1}\big|_{\alpha,\delta}\\
&\qquad+\big|\mathcal{D}_R^-\big(\theta_{i-1}\omega,u^1_{i-1}\big)(\cdot)-\mathcal{D}_R^-\big(\theta_{i-1}\omega,u^2_{i-1}\big)(\cdot)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\quad\leqslant\sum_{k=-\infty}^{i}e^{-\mu(i-k-1)}K\big(e^{-\kappa}C_SC_\delta+1\big)e^{-\kappa(k-1)}\big|u^1_{k-1}-u^2_{k-1}\big|_{\alpha,\delta}\leqslant\frac14\big\|U^1-U^2\big\|_{\mathcal{H}_\kappa}.
\end{aligned}$$

Here we used (4.1), that is,

$$\big|S^-(\cdot)v\big|_{\alpha,\delta}\leqslant C_\delta\|v\|_{V_\delta},\quad v\in D\big(\big(-A\pi^-\big)^\delta\big),$$

and inequality (4.7) for $\delta\in[\alpha,1-\alpha)$.

For the other terms in (4.8), we estimate them in a similar fashion

$$\begin{aligned}
&\sum_{i+1}^{k=0}\big|S^+(\cdot+i-1-k)\big(\mathcal{D}_R^+\big(\theta_{k-1}\omega,u^1_{k-1}\big)(1)-\mathcal{D}_R^+\big(\theta_{k-1}\omega,u^2_{k-1}\big)(1)\big)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\qquad+\big|\hat{\mathcal{D}}_R^+\big(\theta_{i-1}\omega,u^1_{i-1}\big)(\cdot)-\hat{\mathcal{D}}_R^+\big(\theta_{i-1}\omega,u^2_{i-1}\big)(\cdot)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\quad\leqslant\sum_{i+1}^{k=0}C_Se^{(\hat\mu-\kappa)(i-k-1)}e^{-\kappa(k-1)}e^{-\kappa}C^+\big\|\mathcal{D}_R^+\big(\theta_{k-1}\omega,u^1_{k-1}\big)(1)-\mathcal{D}_R^+\big(\theta_{k-1}\omega,u^2_{k-1}\big)(1)\big\|\\
&\qquad+\big|\hat{\mathcal{D}}_R^+\big(\theta_{i-1}\omega,u^1_{i-1}\big)(\cdot)-\hat{\mathcal{D}}_R^+\big(\theta_{i-1}\omega,u^2_{i-1}\big)(\cdot)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\quad\leqslant C_S\sum_{i+1}^{k=0}e^{(\hat\mu-\kappa)(i-k-1)}e^{-\kappa(k-1)}Ke^{-\kappa}C^+\big|u^1_{k-1}-u^2_{k-1}\big|_{\alpha,\delta}\\
&\qquad+\big|\hat{\mathcal{D}}_R^+\big(\theta_{i-1}\omega,u^1_{i-1}\big)(\cdot)-\hat{\mathcal{D}}_R^+\big(\theta_{i-1}\omega,u^2_{i-1}\big)(\cdot)\big|_{\alpha,\delta}e^{-\kappa(i-1)}\\
&\quad\leqslant\sum_{i}^{k=0}e^{\mu(i-k-1)}K\big(e^{-\kappa}C_SC^++1\big)e^{-\kappa(k-1)}\big|u^1_{k-1}-u^2_{k-1}\big|_{\alpha,\delta}\leqslant\frac14\big\|U^1-U^2\big\|_{\mathcal{H}_\kappa}.
\end{aligned}$$

Hence the contraction condition is satisfied. This fact, together with (i) and the property that $\mathcal{H}_\kappa^{u_0^+}$ is a complete space, allow us to apply the Banach fixed point theorem.

(iii) Let $\hat{J}(\omega, U, u_0^+)$ be the mapping defined by (4.8) for $U \in \mathcal{H}_\kappa^{u_0^+}$ and $u_0^+ \in V^+$. Then it is straightforward to see that

$$J(\omega, U) = \hat{J}\big(\omega, U, \pi^+ U(-1, 1)\big).$$

By the fixed point property of $\Gamma(u_{0j}^+, \omega)$ for $u_{0j}^+ \in V^+$ for $j = 1, 2$, we have

$$\begin{aligned}
\big\|\Gamma\big(u_{01}^+, \omega\big) - \Gamma\big(u_{02}^+, \omega\big)\big\|_{\mathcal{H}_\kappa} &\leqslant \big\|\hat{J}\big(\omega, \Gamma\big(u_{01}^+, \omega\big), u_{01}^+\big) - \hat{J}\big(\omega, \Gamma\big(u_{02}^+, \omega\big), u_{02}^+\big)\big\|_{\mathcal{H}_\kappa} \\
&\leqslant \big\|\hat{J}\big(\omega, \Gamma\big(u_{01}^+, \omega\big), u_{01}^+\big) - \hat{J}\big(\omega, \Gamma\big(u_{01}^+, \omega\big), u_{02}^+\big)\big\|_{\mathcal{H}_\kappa} \\
&\quad + \big\|\hat{J}\big(\omega, \Gamma\big(u_{01}^+, \omega\big), u_{02}^+\big) - \hat{J}\big(\omega, \Gamma\big(u_{02}^+, \omega\big), u_{02}^+\big)\big\|_{\mathcal{H}_\kappa} \\
&\leqslant \big\|S^+(\cdot + i - 1)\big(u_{01}^+ - u_{02}^+\big)\big\|_{\mathcal{H}_\kappa} \\
&\quad + \big\|\hat{J}\big(\omega, \Gamma\big(u_{01}^+, \omega\big), u_{02}^+\big) - \hat{J}\big(\omega, \Gamma\big(u_{02}^+, \omega\big), u_{02}^+\big)\big\|_{\mathcal{H}_\kappa} \\
&\leqslant e^{\hat{\mu}} C^+ \big\|u_{01}^+ - u_{02}^+\big\| + \frac{1}{2}\big\|\Gamma\big(u_{01}^+, \omega\big) - \Gamma\big(u_{02}^+, \omega\big)\big\|_{\mathcal{H}_\kappa},
\end{aligned}$$

which implies that $\Gamma(u_0^+, \omega)$ is Lipschitz continuous in u_0^+. This completes the proof of the theorem. □

Remark 4.6. A straightforward calculation shows that $\Gamma(u_0^+, \omega)(-1, \cdot) \in W^{\alpha,\delta,\infty}(0, 1; V)$ is a solution of

$$u(t) = S(t)\Gamma\big(u_0^+, \omega\big)(-1, 0) + \mathcal{D}_R(\omega, u)(t)$$

on $[0, 1]$ with $\pi^+ u(1) = u_0^+$.

Let $\varphi_R(\cdot, \omega, v_0)$ be the solution of

$$u = S v_0 + \mathcal{D}_R(\omega, u)$$

on $[0, 1]$ which exists in $W^{\alpha,\delta,\infty}(0, 1; V)$ for $v_0 \in V_\delta$, see (4.1).

The next result gives a relationship between the fixed points of $J(\omega)$ and $J(\theta_{-1}\omega)$.

Lemma 4.7. *For $u_0^+ \in V^+$ let $u_{-1}^+ := \pi^+\Gamma(u_0^+, \omega)(-1, 0)$. Then the unique fixed point $\Gamma(u_0^+, \omega)$ of $J(\omega, U) \in \mathcal{H}_\kappa^{u_0^+}$ can be expressed by*

$$\Xi\big(u_0^+, \omega\big)(i, \cdot) = \begin{cases} \Gamma(u_{-1}^+, \theta_{-1}\omega)(i + 1, \cdot), & i = -2, -3, \ldots, \\ \varphi_R(\cdot, \theta_{-1}\omega, \Gamma(u_{-1}^+, \theta_{-1}\omega)(-1, 1)), & i = -1. \end{cases} \tag{4.10}$$

Proof. Since $\Gamma(u_{-1}^+, \theta_{-1}\omega)$ is the fixed point of $J(\theta_{-1}\omega)$, by using the definition of $J(\theta_{-1}\omega)$, we may write the V^- part of $\Gamma(u_{-1}^+, \theta_{-1}\omega)$ as

$$\begin{aligned}\Gamma^-\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1) &= \sum_{k=-\infty}^{-1} S^-(-k)\mathcal{D}_R^-\big(\theta_{k-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k-1,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,\cdot)\big)(1)\\ &= \sum_{k=-\infty}^{0} S^-(-k)\mathcal{D}_R^-\big(\theta_{k-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k-1,\cdot)\big)(1).\end{aligned}$$

Then, using the definition of $\Xi(u_0^+,\omega)$, we have for the $i=-1$ component

$$\begin{aligned}\Xi^-\big(u_0^+,\omega\big)(-1,t) &= \varphi_R^-\big(t,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\\ &= \sum_{k=-\infty}^{0} S^-(t-k)\mathcal{D}_R^-\big(\theta_{k-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k-1,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)\big)(-1,1)\big)(t)\\ &= \sum_{k=-\infty}^{-1} S^-(t-k-1)\mathcal{D}_R^-\big(\theta_{k-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k+1-1,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)\big)(-1,1)\big)(t)\\ &= \sum_{k=-\infty}^{-1} S^-(t-k-1)\mathcal{D}_R^-\big(\theta_{k-1}\omega,\Xi\big(u_0^+,\omega\big)(k-1,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{-1}\omega,\Xi\big(u_0^+,\omega\big)(-1,\cdot)\big)(t).\end{aligned}\tag{4.11}$$

We now consider the i-th component when $i=-2,-3,\ldots.$ Since $\Gamma(u_{-1}^+,\theta_{-1}\omega)$ is the fixed point of $J(\theta_{-1}\omega)$, we have

$$\begin{aligned}\Gamma^-\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot) &= \sum_{k=-\infty}^{i+1} S^-(t+i+1-k)\mathcal{D}_R^-\big(\theta_{k-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k-1,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{i+1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot)\big)(\cdot)\\ &= \sum_{k=-\infty}^{i} S^-(t+i-k)\mathcal{D}_R^-\big(\theta_{k-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{i}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot)\big)(t).\end{aligned}$$

Thus, by the definition of $\Xi(u_0^+,\omega)$, we have

$$\begin{aligned}\Xi^-\big(u_0^+,\omega\big)(i,t) &= \sum_{k=-\infty}^{i} S^-(t+i-k)\mathcal{D}_R^-\big(\theta_{k-1}\omega,\Xi\big(u_0^+,\omega\big)(k-1,\cdot)\big)(1)\\ &\quad+\mathcal{D}_R^-\big(\theta_{i}\omega,\Xi\big(u_0^+,\omega\big)(i,\cdot)\big)(t).\end{aligned}\tag{4.12}$$

Now, we study the V^+ part of $\Xi(u_0^+,\omega)(i,t)$ for $i=-1$. We have

$$\begin{aligned}\Xi^+\big(u_0^+,\omega\big)(-1,t)&=\varphi_R^+\big(t,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\\&=S^+(t)u_{-1}^++\mathcal{D}_R^+\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\big)(t)\\&=S^+(t-1)\big(S^+(1)u_{-1}^++\mathcal{D}_R^+\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\big)(1)\big)\\&\quad-\hat{\mathcal{D}}_R^+\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\big)(t)\\&=S^+(t-1)u_0^+-\hat{\mathcal{D}}_R^+\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\big)(t)\\&=S^+(t-1)u_0^+-\hat{\mathcal{D}}_R^+\big(\theta_{-1}\omega,\Xi\big(u_0^+,\omega\big)(-1,\cdot)\big)(t).\end{aligned}\tag{4.13}$$

And finally for $i=-2,-3,\dots$, we have

$$\begin{aligned}&\Gamma^+\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot)\\&\quad=S^+(t+i+1)u_{-1}^+-\sum_{i+3}^{k=0}S^+(t+i+1-k)\mathcal{D}_R^+\big(\theta_{k-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k-1,\cdot)\big)(1)\\&\qquad-\hat{\mathcal{D}}_R^+\big(\theta_{i+1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot)\big)(t)\\&\quad=S^+(t+i)\big(S^+(1)u_{-1}^++\mathcal{D}_R^+\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\big)(1)\big)\\&\qquad-\sum_{i+3}^{k=0}S^+(t+i+1-k)\mathcal{D}_R^+\big(\theta_{k-1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(k-1,\cdot)\big)(1)\\&\qquad-S^+(t+i)\mathcal{D}_R^+\big(\theta_{-1}\omega,\varphi_R\big(\cdot,\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(-1,1)\big)\big)(1)\\&\qquad-\hat{\mathcal{D}}_R^+\big(\theta_{i+1}\theta_{-1}\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot)\big)(t).\end{aligned}\tag{4.14}$$

Then on account of (4.10) and Remark 4.6 the right-hand side of the last formula is equal to

$$\begin{aligned}&S^+(t+i)u_0^+-\sum_{i+3}^{k=1}S^+(t+i+1-k)\mathcal{D}_R^+\big(\theta_{k-2}\omega,\Xi\big(u_0^+,\omega\big)(k-2,\cdot)\big)(1)\\&\qquad-\hat{\mathcal{D}}_R^+\big(\theta_i\omega,\Gamma\big(u_{-1}^+,\theta_{-1}\omega\big)(i+1,\cdot)\big)(t)\\&\quad=S^+(t+i)u_0^+-\sum_{i+2}^{k=0}S^+(t+i-k)\mathcal{D}_R^+\big(\theta_{k-1}\omega,\Xi\big(u_0^+,\omega\big)(k-1,\cdot)\big)(1)\\&\qquad-\hat{\mathcal{D}}_R^+\big(\theta_i\omega,\Xi\big(u_0^+,\omega\big)(i,\cdot)\big)(t).\end{aligned}\tag{4.15}$$

Combining (4.11)–(4.15), and (4.10) together gives

$$\begin{aligned}\Xi\big(u_0^+,\omega\big)(i,t)=\sum_{k=-\infty}^{i}&S^-(t+i-k)\mathcal{D}_R^-\big(\theta_{k-1}\omega,\Xi\big(u_0^+,\omega\big)(k-1,\cdot)\big)(1)\\&+\mathcal{D}_R^-\big(\theta_i\omega,\Xi\big(u_0^+,\omega\big)(i,\cdot)\big)(t)\end{aligned}$$

$$-\sum_{i+2}^{k=0} S^+(t+i-k)\mathcal{D}_R^+\big(\theta_{k-1}\omega, \Xi\big(u_0^+,\omega\big)(k-1,\cdot)\big)(1)$$
$$-\hat{\mathcal{D}}_R^+\big(\theta_i\omega, \Xi\big(u_0^+,\omega\big)(i,\cdot)\big)(t) + S^+(t+i)u_0^+,$$

which yields that $\Xi(u_0^+,\omega)$ is the fixed point of $J(\omega)$. By the uniqueness of the fixed point, we have $\Xi(u_0^+,\omega) = \Gamma(u_0^+,\omega)$. This completes the proof of the lemma. □

5. Random unstable manifolds

In this section, we prove the existence of a random unstable invariant manifold. As we mentioned in Section 3, $\hat{\mu}$ can be any positive number less than the smallest positive eigenvalue of A and $\check{\mu}$ can be any negative number larger than the largest negative eigenvalue of A. Here, we choose $\hat{\mu}$ and $\check{\mu}$ such that

$$\kappa = \frac{\hat{\mu}+\check{\mu}}{2} > 0.$$

Let R be the tempered from below random variable defined by $R(\omega) = \min(\tilde{R}(\omega), 1)$ where $\tilde{R}$ is given by (4.6). We consider the ball $B_{V^+}(0, \hat{r}(\omega))$ with

$$\hat{r}(\omega) = \frac{R(\theta_{-1}\omega)}{2L_\Gamma e^{-\kappa}} \tag{5.1}$$

which is also tempered from below. In fact, $R(\theta_{-1}\omega)$ is tempered from below since $\Lambda_\alpha^{0,1}(\theta_{-1}\omega) = \Lambda_\alpha^{-1,0}(\omega)$ (see [15]). One can also prove, in a similar way as in Lemma 3.1, that $\Lambda_\alpha^{-1,0}(\omega)$ is tempered from above.

On account of Theorem 4.5(iii) and the definition of the norm in $\mathcal{H}_k$, we have that for $u_0^+ \in B_{V^+}(0, \hat{r}(\omega))$

$$\big|\Gamma\big(u_0^+,\omega\big)(-1,\cdot)\big|_{\alpha,\delta} \leqslant \frac{R(\theta_{-1}\omega)}{2}.$$

We set

$$M(\omega) = \big\{x^+ + \Gamma^-\big(x^+,\omega\big)(-1,1)\colon x^+ \in B_{V^+}\big(0,\hat{r}(\omega)\big)\big\} \quad \text{and}$$
$$m(\omega,\cdot) = \Gamma^-(\cdot,\omega)(-1,1)\big|_{B_{V^+}(0,\hat{r}(\omega))}.$$

Before we show $M(\omega)$ is a local unstable manifold for the RDS φ with discrete time set $\mathbb{T}^+ = \mathbb{Z}^+$, we need the following lemma.

Lemma 5.1. *Let r be a positive random variable, tempered from below, C be a positive constant. Then there exists a positive random variable ρ, tempered from below, such that*

$$\rho(\omega)Ce^{\kappa(i-1)} \leqslant r(\theta_{i-1}\omega) \quad \textit{for } i \in \mathbb{Z}^-.$$

Proof. Define

$$\rho(\omega) = \inf_{i\in\mathbb{Z}^-} \frac{1}{C} e^{-\kappa(i-1)} r(\theta_{i-1}\omega) > 0. \tag{5.2}$$

Then for $k \in \mathbb{Z}^-$, we have

$$\rho(\theta_k\omega)e^{-\kappa k} = \frac{1}{C}\inf_{i\in\mathbb{Z}^-} e^{-\kappa(i-1+k)} r(\theta_{i-1+k}\omega) = \frac{1}{C}\inf_{i\leqslant k} e^{-\kappa(i-1)} r(\theta_{i-1}\omega).$$

Since r is tempered from below the right-hand side tends to ∞ for $k \to -\infty$, hence ρ is tempered from below. □

We now apply this lemma as follows. Let $r = \hat{r}$ given by (5.1) and $C = L_\Gamma e^{-\kappa}$. Let $\hat{\rho}$ be the corresponding random variable given by (5.2), that is,

$$\hat{\rho}(\omega) = \inf_{i\in\mathbb{Z}^-} \frac{1}{C} e^{-\kappa(i-1)} \hat{r}(\theta_{i-1}\omega). \tag{5.3}$$

Then, for $\|u_0^+\| \leqslant \hat{\rho}(\omega)$, by using Theorem 4.5(iii), we have

$$\hat{r}(\theta_i\omega) \geqslant \hat{\rho}(\omega) L_\Gamma e^{-\kappa} e^{\kappa i} \geqslant \left\|\Gamma^+\left(\omega, u_0^+\right)(i-1,1)\right\|, \quad i \in \mathbb{Z}^-.$$

Thus, by using Lemma 4.7, we have

$$x_i(\omega) := \Gamma\left(u_0^+, \omega\right)(i-1,1) \in M(\theta_i\omega).$$

On the other hand, by iterating the first line in (4.10) we have that

$$\left|\Gamma\left(u_0^+, \omega\right)(i-1,\cdot)\right|_{\alpha,\delta} = \left|\Gamma\left(u_i^+, \theta_i\omega\right)(-1,\cdot)\right|_{\alpha,\delta} \leqslant L_\Gamma e^{-\kappa} \hat{r}(\theta_i\omega) = \frac{R(\theta_{i-1}\omega)}{2} \tag{5.4}$$

where $x_i^+ = u_i^+ := \Gamma^+(u_0^+, \omega)(i-1,1)$. Then for the cut-off function χ_R we have

$$\chi_{R(\theta_{i-1}\omega)}\left(\Gamma\left(u_0^+, \omega\right)(i-1,\cdot)\right) = 1$$

for all $u_0^+ \in B_{V^+}(0, \hat{\rho}(\omega))$ and $i \in \mathbb{Z}^-$. Since $\Gamma(u_0^+, \omega)$ is the fixed point of $J(\omega)$, using the estimate (5.4), $\Gamma(u_0^+, \omega)(i-1,t)$ satisfies the following

$$\begin{aligned}\Gamma\left(u_0^+, \omega\right)(i-1,t) &= S(t)\Gamma\left(u_0^+, \omega\right)(i-1,0) + \mathcal{D}\left(\theta_{i-1}\omega, \Gamma\left(u_0^+, \omega\right)(i-1,\cdot)\right)(t)\\ &= S(t)\Gamma\left(u_0^+, \omega\right)(i-2,1) + \mathcal{D}\left(\theta_{i-1}\omega, \Gamma\left(u_0^+, \omega\right)(i-1,\cdot)\right)(t), \quad t \in [0,1].\end{aligned}$$

Then, using this equation and the cocycle property of φ, we obtain

$$\varphi\left(-i+1, \theta_{i-1}\omega, x_{i-1}(\omega)\right) = \Gamma\left(u_0^+, \omega\right)(-1,1) \in M(\omega). \tag{5.5}$$

The definition of x_{i-1} implies

$$\lim_{i\to-\infty} x_{i-1}(\omega) = 0$$

with exponential rate κ.

We now show (2) in the definition of a local invariant manifold. We define the mapping

$$\begin{aligned}H_j(\omega,\cdot) &:= \Gamma^+(\cdot,\theta_j\omega)(-j-1,1) : B_{V^+}\big(0,\hat\rho(\theta_j\omega)\big) \to \Gamma^+\big(B_{V^+}\big(0,\hat\rho(\theta_j\omega)\big),\theta_j\omega\big)(-j-1,1)\\ &=: S_j(\omega),\end{aligned}$$

$j\in\mathbb{Z}^+$. This mapping is continuous because of Theorem 4.5(iii) and the continuity of the mappings

$$W^{\alpha,\delta,\infty}(0,1;V)\ni u\to u(0)\in V$$

and π^+. On the other hand, the mapping

$$x^+\in S_j(\omega)\to\varphi^+\big(j,\omega,\Gamma\big(x^+,\theta_j\omega\big)(-j-1,1)\big)$$

where the range of this mapping is equal to $B_{V^+}(0,r_j(\omega))$, is continuous such that $H_j(\omega)$ is a homeomorphism. Hence $S_j(\omega)$ is a closed neighborhood of 0 in V^+. Moreover by (5.5)

$$\varphi\big(j,\omega,\Gamma\big(x^+,\theta_j\omega\big)(-j-1,1)\big)\in M(\theta_j\omega)\quad\text{for } j=1,\dots,k,\ \text{if } x^+\in\bigcap_{j=1,\dots,k}S_j(\omega)$$

which is a closed neighborhood of 0 in V^+.

We now consider the case of time set $\mathbb{R}^+$. We first recall that for $u_0^+\in B_{V^+}(0,\hat\rho(\omega))$ it holds

$$\Gamma\big(u_0^+,\omega\big)(i-2,1)=\Gamma\big(u_0^+,\omega\big)(i-1,0)=x_{i-1}(\omega)\in M(\theta_{i-1}\omega)$$

and

$$\varphi_R(\cdot,\theta_{i-1}\omega,x_{i-1})=\varphi(\cdot,\theta_{i-1}\omega,x_{i-1})\quad\text{on } [0,1].$$

Then, from the discrete time case we have

$$\begin{gathered}M(\omega)\ni\Gamma\big(u_0^+,\omega\big)(-1,1)=\varphi\big(-i+1,\theta_{i-1}\omega,x_{i-1}(\omega)\big)=\varphi\big(-i+1-t,\theta_{i-1+t}\omega,x_{i-1+t}(\omega)\big),\\ x_{i-1+t}(\omega):=\Gamma\big(u_0^+,\omega\big)(i-1,t)=\varphi\big(t,\theta_{i-1}\omega,x_{i-1}(\omega)\big),\quad t\in(0,1),\end{gathered}$$

where x_{i-1+t} tends to 0 with exponential order κ for $i\to-\infty$.

We now prove that $x_{i-1+t}(\omega)\in M(\theta_{i-1+t}\omega)$ for sufficiently small u_0^+ and $i\in\mathbb{Z}^-$, $t\in[0,1]$. To this end we will show that

$$\big\|x_{i-1+t}(\omega)\big\|\leqslant\hat r(\theta_{i-1+t}\omega)\tag{5.6}$$

and

$$x_{i-1+t}(\omega)=\Gamma\big(u_{-1+t}^+,\theta_{-1+t}\omega\big)(i-1,1)=\Gamma\big(u_{i-1+t}^+,\theta_{i-1+t}\omega\big)(-1,1)\tag{5.7}$$

for $u_{-1+t}^+:=\Gamma^+(u_0^+,\omega)(-1,t)$ and

$$u_{-2+t}^+:=\Gamma\big(u_{-1+t}^+,\theta_{-1+t}\omega\big)(-1,0),\qquad u_{-3+t}^+:=\Gamma\big(u_{-2+t}^+,\theta_{-2+t}\omega\big)(-1,0),\quad\dots.$$

Note that the last equality in (5.7) is a consequence of (4.10).

The following lemma gives us a random variable $\bar\rho$ such that if $\|u_0^+\| \leqslant e^\kappa L_\Gamma^{-1}\bar\rho(\omega)$, then

$$u^+_{-1+t} \in B_{V^+}\big(0, \hat\rho(\theta_{-1+t}\omega)\big).$$

Hence (5.6) and the representation formula of Remark 4.6 for $\Gamma(u^+_{-1+t}, \theta_{-1+t}\omega)(s)$ hold.

Lemma 5.2. *Define $\bar\rho(\omega)$ by Lemma 5.1 with $C = L_\Gamma e^{-\kappa}$ and $\bar r(\omega) = \inf_{\tau\in[-1,0]} \hat r(\theta_\tau\omega)$. If $u_0^+ \in B_{V^+}(0, \bar\rho(\omega)e^\kappa L_\Gamma^{-1})$, then $u^+_{-1+t} \in B_{V^+}(0, \hat\rho(\theta_{-1+t}\omega))$ for $t \in [0,1]$ and $\hat\rho(\omega)$ given by (5.3).*

Proof. We first note that $\bar r$ is a tempered from below random variable since

$$\sup_{t\in[-1,0]} \Lambda_\alpha^{-1,0}(\theta_t\omega) \leqslant \Lambda_\alpha^{-2,0}(\omega),$$

and the latter term is a tempered from above random variable.

Let us check that $\bar\rho(\omega) \leqslant \hat\rho(\theta_{-1+t}\omega)$ for $t \in [0,1]$:

$$\begin{aligned}\hat\rho(\theta_{-1+t}\omega) &= \inf_{i\in\mathbb{Z}^-} \frac{1}{C} e^{-\kappa(i-1)} \hat r(\theta_{i-1}\theta_{-1+t}\omega) \geqslant \inf_{i\in\mathbb{Z}^-} \frac{1}{C} e^{-\kappa(i-1)} \inf_{t\in[0,1]} \hat r(\theta_{i-1}\theta_{-1+t}\omega)\\ &= \inf_{i\in\mathbb{Z}^-} \frac{1}{C} e^{-\kappa(i-1)} \inf_{\tau\in[-1,0]} \hat r(\theta_{i-1}\theta_\tau\omega) = \inf_{i\in\mathbb{Z}^-} \frac{1}{C} e^{-\kappa(i-1)} \bar r(\theta_{i-1}\omega) = \bar\rho(\omega).\end{aligned}$$

Then, applying again Theorem 4.5(iii), we get

$$\big\|u^+_{-1+t}\big\| \leqslant e^{-\kappa} L_\Gamma \big\|u_0^+\big\| \leqslant e^{-\kappa} L_\Gamma e^\kappa L_\Gamma^{-1} \bar\rho(\omega) \leqslant \hat\rho(\theta_{-1+t}\omega). \qquad \square$$

For $\|u_0^+\| \leqslant \hat\rho(\omega)$ the term $x_{i-1}(\omega)$ can be represented by the formula in Remark 4.3 for $U(\tau) = \Gamma(u_0^+, \omega)(\tau)$ and $s = i-1$. Similarly, if $u^+_{-1+t} \in B_{V^+}(0, \hat\rho(\theta_{-1+t}\omega))$ for $t \in [0,1]$, then $\Gamma(u^+_{-1+t}, \theta_{-1+t}\omega)$ is given by

$$\begin{aligned}\Gamma\big(u^+_{-1+t}, \theta_{-1+t}\omega\big)(s) = &\int_{-\infty}^{s} S^-(s-\tau) G\big(\Gamma\big(u^+_{-1+t}, \theta_{-1+t}\omega\big)(\tau)\big)\, d\theta_{-1+t}\omega(\tau)\\ &+ \int_{-\infty}^{s} S^-(s-\tau) F\big(\Gamma\big(u^+_{-1+t}, \theta_{-1+t}\omega\big)(\tau)\big)\, d\tau\\ &+ S^+(s) u^+_{-1+t} + \int_0^s S^+(s-\tau) G\big(\Gamma\big(u^+_{-1+t}, \theta_{-1+t}\omega\big)(\tau)\big)\, d\theta_{-1+t}\omega(\tau)\\ &+ \int_0^s S^+(s-\tau) F\big(\Gamma\big(u^+_{-1+t}, \theta_{-1+t}\omega\big)(\tau)\big)\, d\tau.\end{aligned}$$

Generalizing the proof of Lemma 4.7 to continuous time case yields that

$$\begin{aligned}x_{i-1+t}(\omega) &= \varphi\big(t, \theta_{i-1}\omega, x_{i-1}(\omega)\big) = \varphi\big(t, \theta_{i-1}\omega, \Gamma\big(u^+_{i-1}, \theta_{i-1}\omega\big)(-1,1)\big)\\ &= \Gamma\big(u^+_{i-1+t}, \theta_{i-1+t}\omega\big)(-1,1).\end{aligned}$$

We refer also this relation to Lu and Schmalfuß [21, Lemma 5.5]. Hence $x_{i-1+t}(\omega) \in M(\theta_{i-1+t}\omega)$ for sufficiently small u^+_{-1+t}.

For the proof of (2) in Definition 2.6 for the continuous time case, we can just use the result from the discrete time case replacing $\hat{\rho}$ by $e^{\kappa} L_\Gamma^{-1} \bar{\rho}$, together with the remark that if

$$\varphi^+(j, \omega, u_0) \in B_{V^+}\big(0, e^{\kappa} L_\Gamma^{-1} \bar{\rho}(\theta_j \omega)\big)$$

then $\varphi^+(t, \omega, u_0) \in M(\theta_t \omega)$ for $0 \leqslant t \leqslant j$.

6. Applications to parabolic PDEs with fractional noise

Consider the $2m$-th order parabolic partial equation

$$\frac{\partial u}{\partial t}(t, \xi) = L_{2m} u(t)(\xi) + \mu u(t)(\xi) + F\big(u(t)\big)(\xi) + G\big(u(t)\big)(\xi) \varXi^H(t)(\xi)$$

on $[0, T] \times \mathcal{O}$ with the initial condition

$$u(0, \xi) = u_0(\xi), \quad \xi \in \mathcal{O},$$

and the homogeneous Dirichlet boundary conditions

$$\frac{\partial^k u}{\partial \nu^k}(t, \xi) = 0, \quad (t, \xi) \in (0, T) \times \partial\mathcal{O},\ k = 0, 1, \ldots, m-1,$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with a smooth boundary,

$$-L_{2m} = \sum_{|\gamma| \leqslant 2m} a_\gamma(\xi) D^\gamma$$

is a uniformly elliptic operator with $a_\gamma \in C^\infty(\bar{\mathcal{O}})$, and $\varXi^H$ is the noise, fractional in time.

We want to write the above system as Eq. (3.7) in the space $V = L^2(\mathcal{O})$. Consider

$$A = L_{2m} + \mu, \qquad \operatorname{Dom}(A) = \left\{\varphi \in H^{2m}(\mathcal{O}),\ \frac{\partial^k}{\partial \nu^k}\varphi = 0 \text{ on } \partial\mathcal{O} \text{ for } k = 0, \ldots, m-1\right\}.$$

It is known that A has a compact resolvent and has countably many eigenvalues μ_j of finite multiplicity, which tend to $-\infty$ as $j \to \infty$. Furthermore, the associated eigenfunctions $\{\hat{e}_j\}_{j \in \mathbb{N}}$ form an orthonormal basis of V. Let the parameter $\mu > 0$ be sufficiently large such that there exists $j^* \in \mathbb{N}$

$$\mu_{j^*+1} \leqslant \check{\mu} < 0 < \hat{\mu} \leqslant \mu_{j^*}.$$

Notice that in this situation the value of constant C_S is just 1.

Let

$$V^+ = \operatorname{span}(\hat{e}_j : \mu_j \geqslant \hat{\mu})$$

and V^- be its orthogonal complement space in V. Then V has an invariant splitting

$$V = V^+ \oplus V^-.$$

We now introduce the non-linear operator G. Let

$$g:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$$

twice differentiable with respect to the first argument, $g(0,\xi)=0$, $D_1g(0,\xi)=0$ where

$$\bigl|D_1g(x,\xi)\bigr|\leqslant l(\xi),\qquad \bigl|D_1^2g(x,\xi)\bigr|\leqslant l'(\xi) \tag{6.1}$$

for every $x\in\mathbb{R}$ and almost every $\xi\in\mathcal{O}$ where $l,l'\in V$. It follows therefore that

$$\bigl|g(x,\xi)\bigr|\leqslant l(\xi)|x|.$$

Lemma 6.1. *We introduce the operator*

$$V\ni u\to G(u)\in L(V)$$

defined by

$$G(u)x[\xi]=\int_{\mathcal{O}}g\bigl(u(\eta),\xi\bigr)x(\eta)\,d\eta,\quad x\in L^\infty(\mathcal{O}),$$

which is for every $u\in V$ a bounded linear operator from V to V:

$$\bigl\|G(u)x\bigr\|\leqslant\|l\|\|u\|\|x\|_{L^\infty(\mathcal{O})}.$$

Then G satisfies (3.10) *and* (3.11)*, so G satisfies the assumptions of Theorem* 3.6 *and hence of Lemma* 4.1.

Proof. Let $e\in L^\infty(\mathcal{O})$. We note that $u\to G(u)e$ is differentiable:

$$\begin{aligned}
&\bigl\|G(u+h)e-G(u)e-G'(u)e[h]\bigr\|^2\\
&\quad=\int_{\mathcal{O}}\biggl(\int_{\mathcal{O}}\bigl(g\bigl(u(\eta)+h(\eta),\xi\bigr)e(\eta)-g\bigl(u(\eta),\xi\bigr)e(\eta)-D_1g\bigl(u(\eta),\xi\bigr)e(\eta)h(\eta)\bigr)\,d\eta\biggr)^2d\xi\\
&\quad\leqslant\int_{\mathcal{O}}\biggl(\int_{\mathcal{O}}\bigl|D_1^2g\bigl(u(\eta),\xi\bigr)e(\eta)\bigr|h^2(\eta)\,d\eta\biggr)^2d\xi\\
&\quad\leqslant\|e\|^2_{L^\infty(\mathcal{O})}\bigl\|l'\bigr\|^2\|h\|^4.
\end{aligned}$$

The first derivative $G'(u)e$ of $G(u)e$ is then given by

$$h\to\int_{\mathcal{O}}D_1g\bigl(u(\eta),\xi\bigr)e(\eta)h(\eta)\,d\eta$$

which is a mapping in $L(V)$, see (6.1). We also note that $G'(u)e$ is Lipschitz continuous:

$$
\begin{aligned}
&\sup_{\|h\|=1} \left\| G'(u_1)eh - G'(u_2)eh \right\|^2 \\
&\quad = \sup_{\|h\|=1} \int_{\mathcal{O}} \left(\int_{\mathcal{O}} \left(D_1 g\big(u_1(\eta), \xi\big) e(\eta) h(\eta) - D_1 g\big(u_2(\eta), \xi\big) e(\eta) h(\eta) \right) d\eta \right)^2 d\xi \\
&\quad \leqslant \sup_{\|h\|=1} \int_{\mathcal{O}} \left(\int_{\mathcal{O}} l'(\xi) \big| u_1(\eta) - u_2(\eta) \big| e(\eta) h(\eta)\, d\eta \right)^2 d\xi \\
&\quad \leqslant \|e\|^2_{L^\infty(\mathcal{O})} \left\| l' \right\|^2 \|u_1 - u_2\|^2.
\end{aligned}
$$

We also obtain that $u \to G(u) \in L(V)$ is Lipschitz continuous which follows by the first inequality in (6.1).

It is known that the orthonormal system $\{e_i\}_{i\in\mathbb{N}}$ creating the infinite dimensional fBm can be chosen so that

$$
\sup_{i\in\mathbb{N}} \|e_i\|_{L^\infty(\mathcal{O})} \leqslant N,
$$

see [24]. Hence we can define the Lipschitz constants $L_G = 2N\|l\|$ and $L'_G = 2N\|l'\|$. $\square$

We now consider the nonlinear part F which is supposed to be a Nemytskii operator:

Lemma 6.2. *Let $f : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ such that $f(\cdot, \xi)$ is Lipschitz continuous uniformly for $\xi \in \mathcal{O}$. Then $F(u)(\xi) := f(u(\xi), \xi)$ is Lipschitz continuous on V.*

The proof is straightforward.

Now we formulate conditions on f that ensure the existence of an unstable manifold.

Lemma 6.3. *Let $f : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ satisfying the assumptions of Lemma 6.2 such that $f(\cdot, \xi)$ is twice continuously differentiable uniformly for $\xi \in \mathcal{O}$ and $f(0, \xi) = D_1 f(0, \xi) = 0$. Let χ_R be defined in (4.3). Then F defined in Lemma 6.2 satisfies for $u_1, u_2 \in W^{\alpha,\delta,\infty}(0,1;V)$ and for sufficiently small $\varepsilon > 0$*

$$
\sup_{t\in[0,1]} \left\| F\big(\chi_R(u_1)(t)\big) - F\big(\chi_R(u_2)(t)\big) \right\| \leqslant L_{\mathcal{F}}(R) |u_1 - u_2|_{\alpha,\delta}, \qquad L_{\mathcal{F}}(R) = CR^\varepsilon
$$

for $R > 0$, where C is a positive constant which can be chosen independently of R.

Proof. By the assumptions on f, for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that $|f(u,\xi)| \leqslant C_\varepsilon |u|^{1+\varepsilon}$ for $\xi \in \mathcal{O}$.

Recall that

$$
\chi_R(u) = R \frac{u}{R} q\big(|u|_{\alpha,\delta}/R\big).
$$

Then, for $t \in [0,1]$ we have

$$
\begin{aligned}
&\left\| F\big(u_1(t) q\big(|u_1|_{\alpha,\delta}/R\big)\big) - F\big(u_2(t) q\big(|u_2|_{\alpha,\delta}/R\big)\big) \right\| \\
&\quad \leqslant \left(\int_{\mathcal{O}} \left| f\big(u_1(t,\xi) q\big(|u_1|_{\alpha,\delta}/R\big), \xi\big) - f\big(u_2(t,\xi) q\big(|u_2|_{\alpha,\delta}/R\big), \xi\big) \right|^2 d\xi \right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Big(\int_{\mathcal{O}} \big| f\big(u_1(t,\xi) q\big(|u_1|_{\alpha,\delta}/R\big),\xi\big) - f\big(u_2(t,\xi) q\big(|u_1|_{\alpha,\delta}/R\big),\xi\big)\big|^2 \, d\xi \Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{\mathcal{O}} \big| f\big(u_2(t,\xi) q\big(|u_1|_{\alpha,\delta}/R\big),\xi\big) - f\big(u_2(t,\xi) q\big(|u_2|_{\alpha,\delta}/R\big),\xi\big)\big|^2 \, d\xi \Big)^{\frac{1}{2}} \\
&\leqslant C_\varepsilon \Big(\int_{\mathcal{O}} \big| u_1(t,\xi) - u_2(t,\xi)\big|^{2(1+\epsilon)} \, d\xi \Big)^{\frac{1}{2}} q\big(|u_1|_{\alpha,\delta}/R\big) \\
&\quad + C_\varepsilon \Big(\int_{\mathcal{O}} \big| u_2(t,\xi)\big|^{2(1+\epsilon)} \, d\xi \Big)^{\frac{1}{2}} \big| q\big(|u_1|_{\alpha,\delta}/R\big) - q\big(|u_2|_{\alpha,\delta}/R\big)\big|. \qquad (6.2)
\end{aligned}
$$

Suppose that ε is chosen small enough such that

$$
1 \geqslant \dim(\mathcal{O})\left(\frac{1}{2} - \frac{1}{2(1+\varepsilon)}\right)
$$

such that by the Sobolev embedding theorem $V_\delta \subset H_0^1(\mathcal{O}) \subset L^{2(1+\varepsilon)}(\mathcal{O})$ by Sell and You [33, Lemma 37.8(1)].

Suppose $|u_1|_{\alpha,\delta} \geqslant |u_2|_{\alpha,\delta}$ then $q(|u_1|_{\alpha,\delta}/R), q(|u_2|_{\alpha,\delta}/R) = 0$ for $|u_2|_{\alpha,\delta} \geqslant R$, and $|uq(|u|_{\alpha,\delta}/R)|_{\alpha,\delta} \leqslant R$. Now we can estimate the right-hand side of (6.2) by

$$
\begin{aligned}
&C_\varepsilon \big\| u_1(t) - u_2(t) \big\|_{V_\delta}^{(1+\varepsilon)} q\big(|u_1|_{\alpha,\delta}/R\big) + C_\varepsilon \big\| u_2(t)\big\|_{V_\delta}^{(1+\varepsilon)} \big| q\big(|u_1|_{\alpha,\delta}/R\big) - q\big(|u_2|_{\alpha,\delta}/R\big)\big| \\
&\quad \leqslant C_\varepsilon \left(|u_1 - u_2|_{\alpha,\delta} R^\varepsilon + 2\frac{R^{1+\varepsilon}}{R} |u_1 - u_2|_{\alpha,\delta} \right). \qquad \square
\end{aligned}
$$

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